SHARP UPPER BOUNDS OF THE BETTI NUMBERS FOR A GIVEN HILBERT POLYNOMIAL

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Abstract. We show that there exists a saturated graded ideal in a standard graded polynomial ring which has the largest total Betti numbers among all saturated graded ideals for a fixed Hilbert polynomial.

1. Introduction

A classical problem consists in studying the number of minimal generators of ideals in a local or a graded ring in relation to other invariants of the ring and of the ideals themselves. In particular a great amount of work has been done to establish bounds for the number of generators in terms of certain invariants, for instance: multiplicity, Krull dimension and Hilbert functions (see [M, S]). An important result was proved in [ERV] where the authors established a sharp upper bound for the number of generators $\nu(I)$ of all perfect ideals I in a regular local ring $(R, \mathbf{m}, K)$ (or in a polynomial ring over a field K) in terms of their multiplicity and their height.

In a subsequent paper [V], Valla provides under the same hypotheses sharps upper bounds for every Betti number $\beta_i(I) = \dim_K \operatorname{Tor}_i^R(I, K)$, notice that with this notation $\beta_0(I) = \nu(I)$. More surprisingly Valla proved that among all perfect ideals with a fixed multiplicity and height in a formal power series ring over a field K, there exists one which has the largest possible Betti numbers β_i's.

The main result of this paper is an extension of Valla's Theorem. We will consider both the local and the graded case although the result we present for the local case follows directly from the graded case.

We first consider the graded case. We show that for every fixed Hilbert polynomial $p(t)$, there exist a point Y in the Hilbert scheme $Hilb^{p(t)}_{\mathbb{P}^{n-1}}$ such that $\beta_i(I_Y) \geq \beta_i(I_X)$ for all i and for all $X \in Hilb^{p(t)}_{\mathbb{P}^{n-1}}$. Equivalently, let $S = K[X_1, \dots, X_n]$ be a standard graded polynomial ring over a field K, we prove

Theorem 1.1. *Let $p(t)$ be the Hilbert polynomial of a graded ideal of S. There exists a saturated graded ideal $L \subset S$ with the Hilbert polynomial $p(t)$ such that $\beta_i(S/L) \geq \beta_i(S/I)$ for all i and for all saturated graded ideals $I \subset S$ with the Hilbert polynomial $p(t)$.*

Notice that Valla's result corresponds to the special case of the theorem when $p(t)$ is constant.

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Unfortunately we do not present an explicit formula of the bounds. We are convinced that such a formula, in the general case, would be hard to read and to interpret. Instead, as a part of the proof, we describe the construction of the lex ideal that achieve the bound. Using the Eliahou–Kervaire resolution it is possible to write an explicit formula for the total Betti numbers of every lex ideal in terms of its minimal generators.

In particular explicit computations of the bounds can be carried out for a given Hilbert polynomial. Thus it would be possible to describe an explicit formula of the bounds for classes of simple enough Hilbert polynomials. For example in the special case when the Hilbert polynomials are constant, such a formula was given by Valla [V].

Theorem 1.1 induces the following upper bounds of Betti numbers of ideals in a regular local ring (see Section 3 for the proof). Let $\mathbf{p}_I(t)$ be the Hilbert–Samuel polynomial of an ideal I (see [BH, §4.6]) in a regular local ring $(R, \mathbf{m}, K)$ with respect to $\mathbf{m}$.

Theorem 1.2. *Let $\mathbf{p}(t)$ be the Hilbert–Samuel polynomial of an ideal of a regular local ring $(R, \mathbf{m}, K)$ of dimension n with respect to $\mathbf{m}$. There exists an ideal L in $A = K[[x_1, \dots, x_n]]$ with $\mathbf{p}_L(t) = \mathbf{p}(t)$ such that $\beta_i(A/L) \geq \beta_i(R/I)$ for all i and for all ideals $I \subset R$ with $\mathbf{p}_I(t) = \mathbf{p}(t)$.*

Unfortunately, the proof of Theorem 1.1 is very long and complicated. Moreover, a construction of ideals which achieve the bound is not easy to understand. Thus it would be desirable to get a simpler proof of the theorem and to get a better understanding for the structure of ideals which attain maximal Betti numbers.

The paper is structured in the following way: In Section 2 and 3, we reduce a problem of Betti numbers to a problem of combinatorics of lexicographic sets of monomials with a special structure. In Section 4, we introduce key techniques to prove the main result. In particular, we give a new proof of Valla's result in this section. In Section 5, a construction of ideals which attain maximal Betti numbers of saturated graded ideals for a fixed Hilbert polynomial will be given. In Section 6, we give a proof of the main combinatorial result about lexicographic sets of monomials which essentially proves Theorem 1.1. In Section 7, some examples of ideals with maximal Betti numbers are given.

2. Universal Lex Ideals

In this section, we introduce basic notations which are used in the paper.

Let $S = K[x_1, \dots, x_n]$ be a standard graded polynomial ring over a field K. Let M be a finitely generated graded S-module. The *Hilbert function* $H(M, -) : \mathbb{Z} \to \mathbb{Z}$ of M is the numerical function defined by

$$H(M, k) = \dim_K M_k$$

for all $k \in \mathbb{Z}$, where M_k is the graded component of M of degree k. We denote $P_M(t)$ by the Hilbert polynomial of M. Thus $P_M(t)$ is a polynomial in t satisfying

$P_M(k) = H(M,k)$ for $k \gg 0$. The numbers

$$\beta_{i,j}(M) = \dim_K \mathrm{Tor}_i(M,K)_j$$

are called the *graded Betti numbers* of M, and $\beta_i(M) = \sum_{j\in\mathbb{Z}} \beta_{i,j}(M)$ are called the *(total) Betti numbers* of M.

A set of monomials $W \subset S$ is said to be *lex* if, for all monomials $u \in W$ and $v >_{\mathrm{lex}} u$ of the same degree, one has $v \in W$, where $>_{\mathrm{lex}}$ is the lexicographic order induced by the ordering $x_1 >_{\mathrm{lex}} \cdots >_{\mathrm{lex}} x_n$. A monomial ideal $I \subset S$ is said to be *lex* if the set of monomials in I is lex. By the classical Macaulay's theorem [M], for any graded ideal $I \subset S$ there exists the unique lex ideal $L \subset S$ with the same Hilbert function as I. Moreover, Bigatti [B], Hulett [H] and Pardue [P] proved that lex ideals have the largest graded Betti numbers among all graded ideals having the same Hilbert function.

For any graded ideal $I \subset S$, let

$$\mathrm{sat}\, I = (I : \mathfrak{m}^\infty)$$

be the *saturation* of $I \subset S$, where $\mathfrak{m} = (x_1,\dots,x_n)$ is the graded maximal ideal of S. A graded ideal I is said to be *saturated* if $I = \mathrm{sat}\, I$. It is well-known that I is saturated if and only if $\mathrm{depth}(S/I) > 0$ or $I = S$.

Let $L \subset S$ be a lex ideal. Then $\mathrm{sat}\, L$ is also a lex ideal. It is natural to ask which lex ideals are saturated. The theory of universal lex ideals gives an answer.

A lex ideal $L \subset S$ is said to be *universal* if $LS[x_{n+1}]$ is also a lex ideal in $S[x_{n+1}]$. The followings are fundamental results on universal lex ideals.

Lemma 2.1 ([MH])**.** *Let $L \subset S$ be a lex ideal. The following conditions are equivalent:*

(i) *L is universal;*
(ii) *L is generated by at most n monomials;*
(iii) *$L = S$ or there exist integers $a_1, a_2, \dots, a_t \geq 0$ with $1 \leq t \leq n$ such that*

$$L = (x_1^{a_1+1}, x_1^{a_1}x_2^{a_2+1}, \dots, x_1^{a_1}x_2^{a_2}\cdots x_{t-1}^{a_{t-1}}x_t^{a_t+1}). \tag{1}$$

A relation between universal lex ideals and saturated lex ideals is the following.

Lemma 2.2 ([MH])**.** *Let $L \subsetneq S$ be a lex ideal. Then $\mathrm{depth}(S/L) > 0$ if and only if L is generated by at most $n-1$ monomials.*

A lex ideal $I \subset S$ is called a *proper universal lex ideal* if I is generated by at most $n-1$ monomials or $I = S$.

Let $I \subset S$ be a graded ideal. Then there exists the unique lex ideal $L \subset S$ with the same Hilbert function as I. Then $\mathrm{sat}\, L$ is a proper universal lex ideal with the same Hilbert polynomial as I. This construction $I \to \mathrm{sat}\, L$ gives a one-to-one correspondence between Hilbert polynomials of graded ideals and proper universal lex ideals, say,

Proposition 2.3. *For any graded ideal $I \subset S$ there exists the unique proper universal lex ideal $L \subset S$ with the same Hilbert polynomial as I.*

Proof. The existence is obvious. What we must prove is that, if L and L' are proper universal lex ideals with the same Hilbert polynomial then $L = L'$.

Since L and L' have the same Hilbert polynomial, their Hilbert function coincide in sufficiently large degrees. This fact shows $L_d = L'_d$ for $d \gg 0$. Thus $\operatorname{sat} L = \operatorname{sat} L'$. Since L and L' are saturated, $L = \operatorname{sat} L = \operatorname{sat} L' = L$. $\square$

3. Strongly stable ideals, Betti numbers and max sequences

In this section, we reduce a problem of Betti numbers of graded ideals to a problem of combinatorics of lex sets of monomials.

Let $S = K[x_1, \dots, x_n]$ and $\bar{S} = K[x_1, \dots, x_{n-1}]$. For a monomial ideal $I \subset S$, let $\bar{I} = I \cap \bar{S}$. A monomial ideal $I \subset S$ is said to be *strongly stable* if $ux_j \in I$ and $i < j$ imply $ux_i \in I$. The following fact easily follows from the Bigatti-Hulett-Pardue theorem [B, H, P]. See e.g., the proof of [MH, Theorem 2.1].

Lemma 3.1. *For any saturated graded ideal $I \subset S$, there exists a saturated strongly stable ideal $J \subset S$ with the same Hilbert function as I such that $\beta_{i,j}(I) \leq \beta_{i,j}(J)$ for all i, j. Moreover, we may take J so that $\bar{J}$ is a lex ideal in $\bar{S}$.*

Lemma 3.2. *Let $J \subset S$ be a saturated strongly stable ideal. Then,*

(i) $\dim_K J_d = \sum_{k=0}^{d} \dim_K \bar{J}_k$ *for all $d \geq 0$.*
(ii) $\beta_i^S(J) = \beta_i^{\bar{S}}(\bar{J})$ *for all i.*

Proof. If a strongly stable ideal $J \subset S$ is saturated then x_n is regular on S/J. Then $J = \bar{J}S$, which proves (ii). Also, for all $d \geq 0$, we have a decomposition $J_d = \bigoplus_{k=0}^{d} J_k x_n^{d-k}$ as K-vector spaces. This equality proves (i). $\square$

Corollary 3.3. *Let J and J' be saturated strongly stable ideals in S such that $\bar{J}$ and $\bar{J}'$ are lex. If J and J' have the same Hilbert polynomial then $\bar{J}_d = \bar{J}'_d$ for $d \gg 0$.*

Proof. Lemma 3.2(i) says that $\dim_K J_d - \dim_K J_{d-1} = \dim \bar{J}_d$, so $\dim_K \bar{J}_d = \dim_K \bar{J}'_d$ for $d \gg 0$. Then the statement follows since $\bar{J}$ and $\bar{J}'$ are lex. $\square$

Next, we describe all saturated strongly stable ideals J such that $\bar{J}$ is lex. By Proposition 2.3, to fix a Hilbert polynomial is equivalent to fix a proper universal lex ideal U. For a proper universal lex ideal $U \subset S$, let

$$\begin{aligned} &\mathcal{L}(U) \\ &= \{I \subset \bar{S} : I \text{ is a lex ideal with } I \subset \operatorname{sat} \bar{U} \text{ and } \dim_K (\operatorname{sat} \bar{U})/I = \dim_K (\operatorname{sat} \bar{U})/\bar{U}\}. \end{aligned}$$

Note that $\dim_K(\operatorname{sat} J)/J$ is finite for any graded ideal $J \subset S$ since $(\operatorname{sat} J)/J$ is isomorphic to the 0th local cohomology module $H^0_{\mathfrak{m}}(S/J)$. By using Lemma 3.2, it is easy to see that if $I \in \mathcal{L}(U)$ then IS has the same Hilbert polynomial as U. Actually, the converse is also true.

Lemma 3.4. *Let U be a proper universal lex ideal. If J is a saturated strongly stable ideal such that $\bar{J}$ is lex and $P_J(t) = P_U(t)$, then $\bar{J} \in \mathcal{L}(U)$.*

Proof. By Corollary 3.3 we have $\bar{U}_d = \bar{J}_d$ for $d \gg 0$, so $\operatorname{sat} \bar{U} = \operatorname{sat} \bar{J}$. Also, since U and J have the same Hilbert polynomial, for $d \gg 0$, one has

$$\dim_K U_d = \sum_{k=0}^{d} \dim_K \bar{U}_k = \sum_{k=0}^{d} \dim_K(\operatorname{sat} \bar{U}_k) - \dim_K(\operatorname{sat} \bar{U}/\bar{U})$$

and

$$\dim_K J_d = \sum_{k=0}^{d} \dim_K \bar{J}_k = \sum_{k=0}^{d} \dim_K(\operatorname{sat} \bar{J}_k) - \dim_K(\operatorname{sat} \bar{J}/\bar{J}).$$

Since $\operatorname{sat} \bar{J} = \operatorname{sat} \bar{U}$, we have $\dim_K(\operatorname{sat} \bar{J}/\bar{J}) = \dim_K(\operatorname{sat} \bar{U}/\bar{U})$ and $\bar{J} \in \mathcal{L}(U)$. $\square$

By Lemmas 3.1 and 3.4, to prove Theorem 1.1, it is enough to find a lex ideal which has the largest Betti numbers among all ideals in $\mathcal{L}(U)$. We consider a more general setting. For any universal lex ideal $U \subset S$ (not necessary proper) and for any positive integer $c > 0$, define

$$\mathcal{L}(U; c) = \{I \subset U : I \text{ is a lex ideal with } \dim_K U/I = c\}.$$

We consider the Betti numbers of ideals in $L(U; c)$.

We first discuss Betti numbers of strongly stable ideals. We need the following notation. For any monomial $u \in S$, let $\max u$ be the largest integer ℓ such that x_ℓ divides u, where $\max(1) = 1$. For a set of monomials (or a K-vector space spanned by monomials) M, let

$$m_{\leq i}(M) = \#\{u \in M : \max u \leq i\}$$

for $i = 1, 2, \dots, n$, where $\#X$ is the cardinality of a finite set X, and

$$m(M) = \big(m_{\leq 1}(M), m_{\leq 2}(M), \dots, m_{\leq n}(M)\big).$$

These numbers are often used to study Betti numbers of strongly stable ideals. The next formula was proved by Bigatti [B] and Hulett [H], by using the famous Eliahou–Kervaire resolution [EK].

Lemma 3.5. *Let $I \subset S$ be a strongly stable ideal. Then, for all i, j,*

$$\beta_{i,i+j}(I) = \binom{n-1}{i} \dim_K I_j - \sum_{k=1}^{n} \binom{k-1}{i} m_{\leq k}(I_{j-1}) - \sum_{k=1}^{n-1} \binom{k-1}{i-1} m_{\leq k}(I_j).$$

For vectors $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{Z}^n$, we define

$$\mathbf{a} \succeq \mathbf{b} \quad \Leftrightarrow \quad a_i \geq b_i \text{ for } i = 1, 2, \dots, n.$$

Corollary 3.6. *Let U be a universal lex ideal and $I, J \in \mathcal{L}(U; c)$. Let $\mathcal{M}_I$ (resp. $\mathcal{M}_J$) be the set of all monomials in $U \setminus I$ (resp. $U \setminus J$). If $m(\mathcal{M}_I) \succeq m(\mathcal{M}_J)$ then $\beta_i(I) \geq \beta_i(J)$ for all i.*

Proof. Observe that $\beta_{i,i+j}(I) = \beta_{i,i+j}(J) = 0$ for $j \gg 0$. Thus, for $d \gg 0$, we have $\beta_i(I) = \sum_{j=0}^{d} \beta_{i,i+j}(I)$. Let $I_{\le d} = \bigoplus_{k=0}^{d} I_k$. Then by Lemma 3.5,

$$\beta_i(I) = \binom{n-1}{i} \dim_K I_{\le d} - \sum_{k=1}^{n} \binom{k-1}{i} m_{\le k}(I_{\le d-1}) - \sum_{k=1}^{n-1} \binom{k-1}{i-1} m_{\le k}(I_{\le d})$$

and the same formula holds for J. Since, for $d \gg 0$,

$$m(J_{\le d}) = m(U_{\le d}) - m(\mathcal{M}_J) \succeq m(U_{\le d}) - m(\mathcal{M}_I) = m(I_{\le d}),$$

we have $\beta_i(I) \ge \beta_i(J)$ for all i, as desired. □

Next, we study the structure of $\mathcal{M}_I$. Let

$$U = (x_1^{a_1+1}, x_1^{a_1} x_2^{a_2+1}, \dots, x_1^{a_1} x_2^{a_2} \cdots x_{t-1}^{a_{t-1}} x_t^{a_t+1})$$

be a universal lex ideal, $\delta_i = x_1^{a_1} \cdots x_{i-1}^{a_{i-1}} x_i^{a_i+1}$ and $b_i = a_1 + \cdots + a_i + 1 = \deg \delta_i$. (If $U = S$ then $t = 1$ and $a_1 = -1$.) Let

$$S^{(i)} = K[x_i, \dots, x_n].$$

Then, as K-vector spaces, we have a decomposition

$$U = \delta_1 S^{(1)} \bigoplus \delta_2 S^{(2)} \bigoplus \cdots \bigoplus \delta_t S^{(t)}.$$

Definition 3.7. A set of monomials $N \subset S^{(i)}$ is said to be *rev-lex* if, for all monomials $u \in N$ and $v <_{\text{lex}} u$ of the same degree, one has $v \in N$. Moreover, N is said to be *super rev-lex* (in $S^{(i)}$) if it is rev-lex and $u \in N$ implies $v \in N$ for any monomial $v \in S^{(i)}$ of degree $\le \deg u - 1$. A *multicomplex* is a set of monomials $N \subset S^{(i)}$ satisfying that $u \in N$ and $v | u$ imply $v \in N$. Thus a multicomplex is the complement of the set of monomials in a monomial ideal. Note that super rev-lex sets are multicomplexes.

Let $I \in \mathcal{L}(U; c)$ and $\mathcal{M}_I$ the set of monomials in $U \setminus I$. Then we can uniquely write

$$\mathcal{M}_I = \delta_1 M_{\langle 1 \rangle} \biguplus \delta_2 M_{\langle 2 \rangle} \biguplus \cdots \biguplus \delta_t M_{\langle t \rangle}$$

where $M_{\langle i \rangle} \subset S^{(i)}$ and where $\biguplus$ denotes the disjoint union. The following fact is obvious.

Lemma 3.8. *With the same notation as above,*

(i) *each $M_{\langle i \rangle}$ is a rev-lex multicomplex.*
(ii) *if $\delta_i M_{\langle i \rangle}$ has a monomial of degree d then $\delta_{i+1} M_{\langle i+1 \rangle}$ contains all monomials of degree d in $\delta_{i+1} S^{(i+1)}$ for all d.*

Note that Lemma 3.8(ii) is equivalent to saying that if $M_{\langle i \rangle}$ contains a monomial of degree d then $M_{\langle i+1 \rangle}$ contains all monomials of degree $d - a_{i+1}$ in $S^{(i+1)}$.

We say that a set of monomials

$$M = \delta_1 M_{\langle 1 \rangle} \biguplus \delta_2 M_{\langle 2 \rangle} \biguplus \cdots \biguplus \delta_t M_{\langle t \rangle} \subset U,$$

where $M_{\langle i\rangle} \subset S^{(i)}$, is a *ladder set* if it satisfies the conditions (i) and (ii) of Lemma 3.8. The next result is the key result in this paper.

Proposition 3.9. *Let $U \subset S$ be a universal lex ideal. For any integer $c \geq 0$, there exists a ladder set $N \subset U$ with $\#N = c$ such that for any ladder set $M \subset U$ with $\#M = c$ one has*

$$m(N) \succeq m(M).$$

We prove Proposition 3.9 in Section 6. Here, we prove Theorem 1.1 by using Proposition 3.9.

Proof of Theorem 1.1. Let $U \subset S$ be a proper universal lex ideal with $P_U(t) = p(t)$ and $\bar{U} = U \cap \bar{S}$. Let $c = \dim_K(\operatorname{sat}\bar{U}/\bar{U})$. For any lex ideal $I \subset \operatorname{sat}\bar{U}$, let $\mathcal{M}_I$ be the set of monomials in $(\operatorname{sat}\bar{U} \setminus I)$.

Let $N \subset \operatorname{sat}\bar{U}$ be a ladder set of monomials with $\#N = c$ given in Proposition 3.9. Consider the ideal $J \subset \bar{S}$ generated by all monomials in $\operatorname{sat}\bar{U} \setminus N$. Then $J \subset \operatorname{sat}\bar{U}$ and $\mathcal{M}_J = N$. In particular, $J \in \mathcal{L}(U)$.

Let $L = JS$. By the construction, $P_L(t) = P_U(t) = p(t)$. We claim that L satisfies the desired conditions. Let $I \subset S$ be a saturated graded ideal with $P_I(t) = p(t)$. By Lemmas 3.1 and 3.4, we may assume that I is a saturated strongly stable ideal with $\bar{I} \in \mathcal{L}(U) = \mathcal{L}(\operatorname{sat}\bar{U}; c)$. Since $\mathcal{M}_{\bar{I}}$ is a ladder set, by the choice of J, $m(\mathcal{M}_J) \succeq m(\mathcal{M}_{\bar{I}})$. Then, by Corollary 3.6, $\beta_i(L) = \beta_i(J) \geq \beta_i(\bar{I}) = \beta_i(I)$ for all i, as desired. □

Another interesting corollary of Proposition 3.9 is

Corollary 3.10. *Let $U \subset S$ be a universal lex ideal and $c \geq 0$. There exists a lex ideal $L \subset U$ with $\dim_K U/L = c$ such that, for any graded ideal $I \subset U$ with $\dim_K U/I = c$, one has $\beta_i(L) \geq \beta_i(I)$ for all i.*

Finally we prove Theorem 1.2.

Proof of Theorem 1.2. Let I be an ideal in a regular local ring $(R, \mathbf{m}, K)$ with the Hilbert–Samuel polynomial $\mathbf{p}(t)$. Then the associated graded ring $\operatorname{gr}_{\mathbf{m}}(R/I)$ has the same Hilbert–Samuel polynomial as R/I and $\beta_i(R/I) \leq \beta_i(\operatorname{gr}_{\mathbf{m}}(R/I))$ for all i (see [R] and [HRV]).

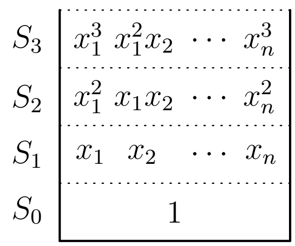
Let $S = K[x_1, \dots, x_n]$ and $S' = S[x_{n+1}]$ be standard graded polynomial rings. By adjoining a variable to $\operatorname{gr}_{\mathbf{m}}(R/I)$ we obtain a graded ring that is isomorphic to S'/J for a saturated graded ideal $J \subset S'$. Then $\mathbf{p}_{\operatorname{gr}_{\mathbf{m}}(R/I)}(t)$ is equal to the Hilbert polynomial of S'/J and $\beta_i(\operatorname{gr}_{\mathbf{m}}(R/I)) = \beta_i(S'/J)$ for all i. Let $L' \subset S'$ be the saturated ideal with the same Hilbert polynomial as J given in Theorem 1.1. Observe that L' has no generators which are divisible by x_{n+1} by the construction given in the proof of Theorem 1.1.

Let $L \subset A = K[[x_1, \dots, x_n]]$ be a monomial ideal having the same generators as L'. We claim that L satisfies the desired conditions. By the construction, the Hilbert–Samuel polynomial of L is equal to the Hilbert polynomial of L' and $\beta_i(L) = \beta_i(L')$ for all i. Since $\beta_i(R/I) \leq \beta_i(S'/J) \leq \beta_i(S'/L')$ and $\mathbf{p}_{R/I}(t) = P_{S'/J}(t) = P_{S'/L'}(t)$, the ideal L satisfies the desired conditions. □

4. Some tools to study max sequence

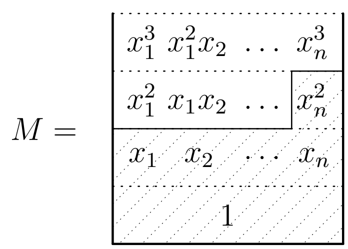
In this section, we introduce some tools to study $m(-)$. Let $S = K[x_1, \ldots, x_n]$ and $\hat{S} = K[x_2, \ldots, x_n]$. From now on, we identify vector spaces spanned by monomials (such as polynomial rings and monomial ideals) with the set of monomials in the spaces. First, we introduce pictures which help to understand the proofs. We associate with the set of monomials in S the following picture in Figure 1.

Figure 1



Each block in Figure 1 represents a set of monomials in S of a fixed degree ordered by the lex order. We represent a set of monomials $M \subset S$ by a shaded picture so that the set of monomials in the shade is equal to M. For example, Figure 2 represents the set $M = \{1, x_1, x_2, \ldots, x_n, x_n^2\}$.

Figure 2



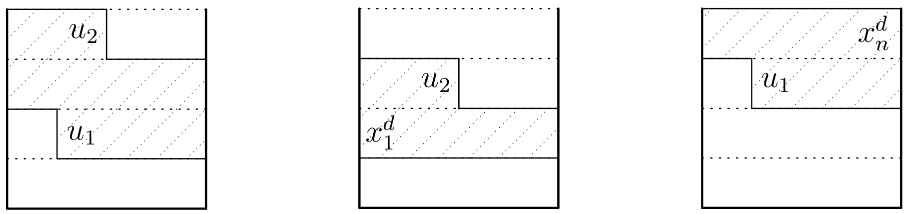
Definition 4.1. We define the *opposite degree lex order* $>_{\text{opdlex}}$ by $u >_{\text{opdlex}} v$ if (i) $\deg u < \deg v$ or (ii) $\deg u = \deg v$ and $u >_{\text{lex}} v$.

For monomials $u_1 \geq_{\text{opdlex}} u_2$, let

$$[u_1, u_2] = \{v \in S : u_1 \geq_{\text{opdlex}} v \geq_{\text{opdlex}} u_2\}.$$

A set of monomials $M \subset S$ is called an *interval* if $M = [u_1, u_2]$ for some monomials $u_1, u_2 \in S$. Moreover, we say that M is a *lower lex set of degree d* if $M = [x_1^d, u_2]$, and that M is an *upper rev-lex set of degree d* if $M = [u_1, x_n^d]$. (See Fig. 3.)

Figure 3



Interval | Lower lex set | Upper rev-lex set

A benefit of considering pictures is that we can visualize the following map $\rho : S \to \hat{S}$. For any monomial $x_1^k u \in S$ with $u \in \hat{S}$, let

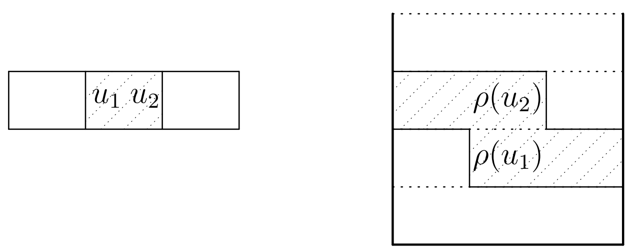
$$\rho(x_1^k u) = u.$$

This induces a bijection

$$\begin{array}{cccc} \rho: & S_d = \bigoplus_{k=0}^{d} x_1^k \hat{S}_{d-k} & \longrightarrow & \hat{S}_{\leq d} = \bigoplus_{k=0}^{d} \hat{S}_k \\ & x_1^k u & \longrightarrow & u. \end{array}$$

It is easy to see that if $[u_1, u_2] \subset S_d$ then $\rho([u_1, u_2]) = [\rho(u_1), \rho(u_2)]$ is an interval in $\hat{S}$. (See Fig. 4.)

Figure 4



$[u_1, u_2] \subset S_d$ | $\rho([u_1, u_2]) \subset \hat{S}_{\leq d}$

In particular, we have

Lemma 4.2. *Let $M \subset S_d$ be a set of monomials.*

(i) *If M is lex then $\rho(M)$ is a lower lex set of degree 0 in $\hat{S}$.*
(ii) *If M is rev-lex then $\rho(M)$ is an upper rev-lex set of degree d in $\hat{S}$.*

We define $\max(1) = 1$ in S and $\max(1) = 2$ in $\hat{S}$. For any monomial $u \in S_d$ with $u \neq x_1^d$, one has $\max(u) = \max(\rho(u))$. Hence

Lemma 4.3. *Let $M \subset S_d$ be a set of monomials. One has $m(M) \succeq m(\rho(M))$. Moreover, if $x_1^d \notin M$ then $m(M) = m(\rho(M))$.*

Lemma 4.4 (Interval Lemma)**.** *Let $[u_1, u_2]$ be an interval in S, $0 \leq a \leq \deg u_1$ and $b \geq \deg u_2$. Let $L \subset S$ be the lower lex set of degree a and R the upper rev-lex set of degree b with $\#L = \#R = \#[u_1, u_2]$. Then*

$$m(L) \succeq m\big([u_1, u_2]\big) \succeq m(R).$$

Proof. We use double induction on n and $\#[u_1, u_2]$. The statement is obvious if $n = 1$ or if $\#[u_1, u_2] = 1$. Suppose $n > 1$ and $\#[u_1, u_2] > 1$.

Case 1. We first prove the statement when $[u_1, u_2]$, L and R are contained in a single component S_d for some degree d. We may assume $L \neq [u_1, u_2]$ and $L \neq R$. Then, since $x_1^d \notin [u_1, u_2]$, $m([u_1, u_2]) = m(\rho([u_1, u_2]))$ and $m(R) = m(\rho(R))$. Since $\rho(L) \subset \hat{S}_{\leq d}$ is a lower lex set of degree 0, $\rho([u_1, u_2]) \subset \hat{S}_{\leq d}$ is an interval and $\rho(R) \subset \hat{S}_{\leq d}$ is an upper rev-lex set of degree d in $\hat{S}$, by the induction hypothesis, we have

$$m(L) \succeq m\big(\rho(L)\big) \succeq m\big(\rho\big([u_1, u_2]\big)\big) \succeq m\big(\rho(R)\big) = m(R).$$

Then the statement follows since $m(\rho([u_1, u_2])) = m([u_1, u_2])$.

Case 2. Now we prove the statement in general. We first prove the statement for L. We identify S_i with the set of monomials in S of degree i. Suppose $\#[u_1, u_2] > \#S_a$. Then there exist $u_1', u_2' \in S$ such that

$$[u_1, u_2] = [u_1, u_2'] \biguplus [u_1', u_2]$$

and $\#[u_1, u_2'] = \#S_a$. Let L' be the lower lex set of degree $a+1$ with $\#L' = \#[u_1', u_2]$. By the induction hypothesis, $m(S_a) \succeq m([u_1, u_2'])$ and $m(L') \succeq m([u_1', u_2])$. Thus

$$m\big([u_1, u_2]\big) \preceq m\big(S_a \biguplus L'\big) = m(L).$$

Suppose $\#[u_1, u_2] \leq \#S_a$. Then $L \subset S_a$. Let $d = \deg u_1$ and $A \subset S_d$ the lex set with $\#A = \#[u_1, u_2]$. Then $A = x_1^{d-a} L$. Since $m(A) = m(L)$, what we must prove is

$$m(A) \succeq m\big([u_1, u_2]\big).$$

Since $\#[u_1, u_2] \leq \#S_a \leq \#S_{d+1}$, we have $\deg u_2 \leq d + 1$.

If $\deg u_2 = d$, then $[u_1, u_2] \subset S_d$. Then the desired inequality follows from Case 1. Suppose $\deg u_2 = d + 1$. Then

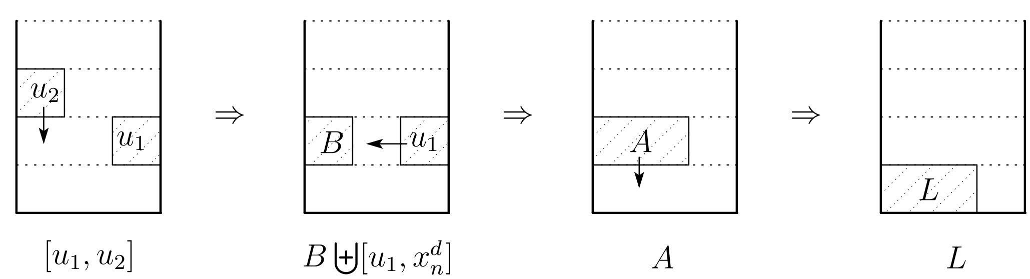
$$[u_1, u_2] = [u_1, x_n^d] \biguplus [x_1^{d+1}, u_2].$$

Recall $\#[u_1, u_2] \leq \#S_a \leq \#S_d$. Let $B \subset S_d$ be the lex set with $\#B = \#[x_1^{d+1}, u_2]$. Then $[x_1^{d+1}, u_2] = x_1 B$. Since $\#B + \#[u_1, x_n^d] = \#[u_1, u_2] \leq \#S_d$, $B \cap [u_1, x_n^d] = \emptyset$. Then, by Case 1,

$$m\big([u_1, u_2]\big) = m(B) + m\big([u_1, x_n^d]\big) \preceq m(A).$$

(See Fig. 5.)

Figure 5



Next, we prove the statement for R. In the same way as in the proof for L, we may assume $\#[u_1, u_2] \leq \#S_b$. Let $d = \deg u_2$.

If $\deg u_1 = d$, then $[u_1, u_2] \subset S_d$ and $A = x_1^{b-d}[u_1, u_2]$ is an interval in S_b. Then, by Case 1, we have $m\big([u_1, u_2]\big) = m(A) \succeq m(R)$ as desired. Suppose $\deg u_1 < d$. Then

$$[u_1, u_2] = [u_1, x_n^{d-1}] \biguplus [x_1^d, u_2].$$

Let R' be the upper rev-lex set of degree b in S with $\#R' = \#[u_1, x_n^{d-1}]$. Then,

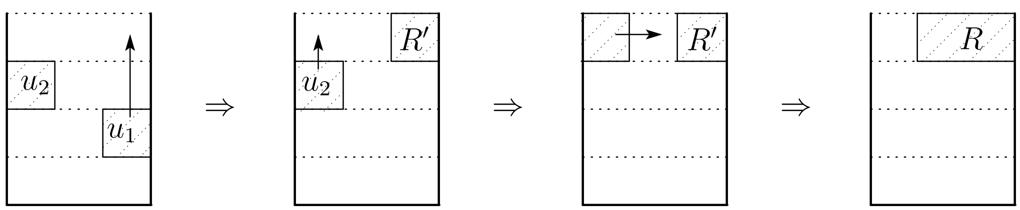
$$m\big([u_1, u_2]\big) \;\; \succeq \;\; m(R') + m\big([x_1^d, u_2]\big) = m(R') + m\big([x_1^b, x_1^{b-d}u_2]\big),$$

where the first inequality follows from the induction hypothesis on the cardinality. Since $R \setminus R' \subset S_b$ is an interval and $[x_1^b, x_1^{b-d}u_2] \subset S_b$ is lex, by Case 1 we have

$$m(R') + m\big([x_1^b, x_1^{b-d}u_2]\big) \succeq m(R') + m(R \setminus R') = m(R),$$

as desired. (See Fig. 6.)

Figure 6



$[u_1, u_2]$ $\qquad$ $R' \biguplus [x_1^d, u_2]$ $\qquad$ $R' \biguplus [x_1^b, x_1^{b-d}u_2]$ $\qquad$ R

$\square$

Recall that a set $M \subset S$ of monomials is said to be super rev-lex if it is rev-lex and $u \in M$ implies $v \in M$ for any monomial $v \in S$ of degree $\leq \deg u - 1$.

Corollary 4.5. *Let $R \subset S$ be an upper rev-lex set of degree d and $M \subset S$ a super rev-lex set such that $\#R + \#M \leq \#S_{\leq d}$. Let $Q \subset S$ be the super rev-lex set with $\#Q = \#R + \#M$. Then*

$$m(Q) \succeq m(R) + m(M).$$

Proof. Let $e = \min\{k : x_1^k \notin M\}$ and $F = \{u \in S_e : u \notin M\}$. If $\#F \geq \#R$ then

$$Q = M \biguplus (Q \setminus M)$$

and $Q \setminus M \subset F$ is an interval. Thus $m(Q \setminus M) \succeq m(R)$ by the interval lemma.

Suppose $\#F < \#R$. Write

$$R = I \biguplus R'$$

such that I is an interval with $\#I = \#F$ and R' is an upper rev-lex set of degree d. Since F is a lex set, the interval lemma shows

$$m(M) + m(R) = m(M) + m(I) + m(R') \preceq m\big(F \biguplus M\big) + m(R').$$

Then $F \biguplus M$ is a super rev-lex set containing x_1^e. By repeating this procedure, we have $m(M) + m(R) \preceq m(Q)$. $\square$

The above corollary proves the next result which was essentially proved in [ERV].

Corollary 4.6 (Elias-Robbiano-Valla). *Let $M \subset S$ be a finite rev-lex set of monomials and $R \subset S$ the super rev-lex set with $\#R = \#M$. Then $m(R) \succeq m(M)$.*

Proof. Let $M = \biguplus_{i=0}^{N} M_i$, where M_i is the set of monomials in M of degree i and $N = \max\{i : M_i \neq \emptyset\}$. Let $R_{(\leq j)}$ be the super rev-lex set with $\#R_{(\leq j)} = \# \biguplus_{i=0}^{j} M_i$. We claim $m(R_{(\leq j)}) \succeq m(\biguplus_{i=0}^{j} M_i)$ for all j. This follows inductively from Corollary 4.5 as follows:

$$m\Big(\biguplus_{i=0}^{j} M_i\Big) = m\Big(\biguplus_{i=0}^{j-1} M_i\Big) + m(M_j) \preceq m(R_{(\leq j-1)}) + m(M_j) \preceq m(R_{(\leq j)}).$$

(We use induction hypothesis for the second step and use Corollary 4.5 for the last step.) Then we have $m(R) = m(R_{(\leq N)}) \succeq m(\biguplus_{i=0}^{N} M_i)$. $\square$

We finish this section by proving the result of Valla which we mentioned in the introduction.

Corollary 4.7 (Valla). *Let c be a positive integer and $M \subset S$ the super rev-lex set with $\#M = c$. Let $J \subset S$ be the monomial ideal generated by all monomials which are not in M. Then, for any homogeneous ideal $I \subset S$ with $\dim_K(S/I) = c$, we have $\beta_i(S/J) \geq \beta_i(S/I)$ for all i.*

Proof. The proof is similar to that of Corollary 3.6. By the Bigatti-Hulett-Pardue theorem, we may assume that I is lex. Then Lemma 3.5 says, for $d \gg 0$, we have

$$\beta_i(I) = \binom{n-1}{i} \dim_K I_{\leq d} - \sum_{k=1}^{n} \binom{k-1}{i} m_{\leq k}(I_{\leq d-1}) - \sum_{k=1}^{n-1} \binom{k-1}{i-1} m_{\leq k}(I_{\leq d})$$

and the same formula holds for J. Let $N \subset S$ be the set of monomials which are not in I. Since N is a rev-lex set with $\#N = c$, for $d \gg 0$, by Corollary 4.6 we have

$$m(J_{\leq d}) = m(S_{\leq d}) - m(M) \preceq m(S_{\leq d}) - m(N) = m(I_{\leq d}).$$

Hence $\beta_i(J) \geq \beta_i(I)$ for all i as desired. $\square$

The proof given in this section provides a new short proof of the above result. The most difficult part in the proof is Corollary 4.6. The original proof given in [ERV] is based on computations of binomial coefficients. On the other hand, our proof is based on moves of interval sets of monomials.

5. Construction

In this section, we give a construction of sets of monomials which satisfies the conditions of Proposition 3.9, and study their properties.

Throughout this section, we fix a universal lex ideal

$$U = (x_1^{a_1+1}, x_1^{a_1} x_2^{a_2+1}, \ldots, x_1^{a_1} x_2^{a_2} \cdots x_{t-1}^{a_{t-1}} x_t^{a_t+1}).$$

We identify vector spaces spanned by monomials (such as polynomial rings and monomial ideals) with the set of monomials in the spaces. Thus, $S^{(i)}$ is the set of

monomials in $K[x_i,\dots,x_n]$ and as we see in Section 3 the universal lex ideal U is identified with

$$U=\delta_1 S^{(1)}\biguplus \delta_2 S^{(2)}\biguplus\cdots\biguplus \delta_t S^{(t)},$$

where $\delta_i=x_1^{a_1}\cdots x_{i-1}^{a_{i-1}}x_i^{a_i+1}$ for $i=1,2,\dots,t$. Let $b_i=\deg\delta_i=a_1+\cdots+a_i+1$ for $i=1,2,\dots,t$..

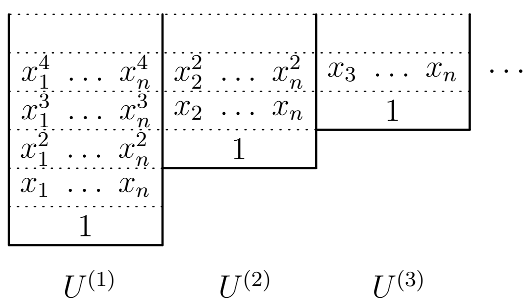
Let $M\subset U$. We write

$$U^{(i)}=\delta_i S^{(i)},\ M^{(i)}=M\cap U^{(i)},\ U^{(\geq i)}=\biguplus_{k=i}^{t}\delta_k S^{(k)} \text{ and } M^{(\geq i)}=M\cap U^{(\geq i)}.$$

Also, we identify $U^{(\geq i)}=\biguplus_{k\geq i}\delta_k S^{(k)}$ with the universal lex ideal in $K[x_i,\dots,x_n]$ generated by $\{\delta'_k=\frac{x_i^{a_1+\cdots+a_{i-1}}}{x_1^{a_1}\cdots x_{i-1}^{a_{i-1}}}\delta_k : k=i,i+1,\dots,t\}$. For any set of monomials M, we write M_k for the set of monomials in M of degree k and $M_{\leq j}=\biguplus_{k=0}^{j}M_k$.

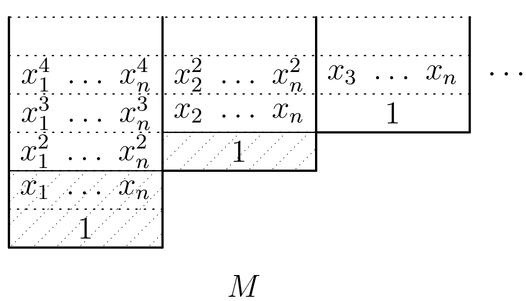
Like Section 4, we use pictures to help to understand the proofs. We identify U with the following picture and present M by a shaded picture.

Figure 7



For example, Figure 8 represents $M=\delta_1\{1,x_1,x_2,\dots,x_n\}\biguplus\delta_2\{1\}$.

Figure 8



Also, we define the map $\rho: U\to U$ by extending the map given in Section 4 as follows: For $\delta_i x_i^k u\in U^{(i)}$ with $u\in K[x_{i+1},\dots,x_n]$, let

$$\rho(\delta_i x_i^k u)=\begin{cases}\delta_{i+1}u, & \text{if } i\leq t-1,\\ 0, & \text{if } i=t.\end{cases}$$

We call the above map $\rho: U\to U$ the *moving map* of U. The moving map induces a bijection from $U_j^{(i)}=\{\delta_i u\in U^{(i)}:\deg u=j-b_i\}$ to $U_{\leq j+a_{i+1}}^{(i+1)}=\{\delta_{i+1}u\in U^{(i+1)}:\deg u\leq j-b_i\}$ for $i=1,2,\dots,t-1$. Also, we have

Lemma 5.1. *For $N \subset U_j^{(i)}$ with $i \leq t-1$, one has $m(N) \succeq m(\rho(N))$. Moreover, if $\delta_i x_i^{j-b_i} \notin N$ then $m(N) = m(\rho(N))$.*

Next, we define ladder sets $M \subset U$ which attain maximal Betti numbers. Recall that a subset $M \subset U$ is called a ladder set if the following conditions holds:

(i) $\{u \in S^{(i)} : \delta_i u \in M^{(i)}\}$ is a rev-lex multicomplex for $i = 1, 2, \ldots, t$.

(ii) if $M_j^{(i)} \neq \emptyset$ then $M_j^{(i+1)} = U_j^{(i+1)}$ for $i = 1, 2, \ldots, t-1$ and for all $j \geq 0$.

To simplify the notation, we say that $N \subset U^{(i)}$ is a super rev-lex set (resp. interval, lower lex set or upper rev-lex set of degree d) if $N' = \{u \in S^{(i)} : \delta_i u \in N\}$ is super rev-lex (resp. interval, lower lex set or upper rev-lex set of degree $d - b_i$) in $S^{(i)}$.

Definition 5.2. A monomial $f = \delta_1 x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \in U_e^{(1)}$ is said to be *admissible over U* if the following conditions hold

(i) $\deg \rho^i(f) \leq e+1$ or $\rho^i(f) = \delta_{i+1}$ for $i = 1, 2, \ldots, t-2$,

(ii) $\rho^{t-1}(f) = \delta_t$ or $\rho^{t-1}(f) \geq_{\mathrm{opdlex}} \delta_t x_t^{e+1-b_t}$.

Note that the second condition in (ii) cannot be satisfied when $e + 1 - b_t < 0$. Also, if $t = 1$ then all monomials in U are admissible. Also $\rho^{t-1}(f) \geq_{\mathrm{opdlex}} \delta_t x_t^{e+1-b_t}$ if and only if $\deg \rho^{t-1}(f) \leq e$ or $\rho^{t-1}(f) = \delta_t x_t^{e+1-b_t}$.

We say that $f \in U_e^{(i)}$ is admissible if it is admissible over $U^{(\geq i)}$. Note that $\delta_i x_i^k \in U^{(i)}$ is admissible for all i and k.

Definition 5.3. Fix $c > 0$. Let $>_{\mathrm{dlex}}$ be the degree lex order. Thus for monomials $u, v \in S$, $u >_{\mathrm{dlex}} v$ if $\deg u > \deg v$ or $\deg u = \deg v$ and $u >_{\mathrm{lex}} v$. Let

$$f = \max_{>_{\mathrm{dlex}}} \{g \in U^{(1)} : \ g \text{ is admissible and } \#\{h \in U : h \leq_{\mathrm{dlex}} g\} \leq c\}$$

and

$$L_{(c)} = \{h \in U^{(1)} : h \leq_{\mathrm{dlex}} f\}.$$

Let $M = M^{(1)} \biguplus \cdots \biguplus M^{(t)} \subset U$ be a set of monomials with $\#M = c$. We say that M satisfies the *maximal condition* if $M^{(1)} = L_{(c)}$. Also, we say that M is *extremal* if $M^{(\geq k)} \subset U^{(\geq k)}$ satisfies the maximal condition in $U^{(\geq k)}$ for all k.

Example 5.4. If $t = 1$ then any monomial in $U = \delta_1 S^{(1)}$ is admissible and extremal sets can be identified with super rev-lex sets in $S^{(1)}$.

Example 5.5. Suppose $t = 2$. Then $f = \delta_1 x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, where $f \neq \delta_1 x_1^{\alpha_1}$, is admissible in $U = \delta_1 S^{(1)} \biguplus \delta_2 S^{(2)}$ if $\alpha_1 \geq a_2$ or $f = \delta_1 x_1^{a_2 - 1} x_2^{\alpha_2}$. In other words, a monomial $f \in \delta_1 S_d^{(1)}$ is admissible if and only if $f \geq_{\mathrm{lex}} \delta_1 x_1^{a_2-1} x_2^{d-a_2+1}$ if $a_2 \leq d$ and $f = \delta_1 x_1^d$ if $a_2 > d$. For example, if $\delta_1 = x_1^2$ and $\delta_2 = x_1 x_2^3$ then the admissible monomials in $U_5^{(1)} = \delta_1(S_3^{(1)})$ are

$$\delta_1 x_1^3, \delta_1 x_1^2 x_2, \delta_1 x_1^2 x_3, \ldots, \delta_1 x_1^2 x_n, \delta_1 x_1 x_2^2.$$

Example 5.6. Suppose $t = 3$. The situation is more complicated. A monomial $f = \delta_1 x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \in U_e^{(1)}$, where $f \neq \delta_1 x_1^{\alpha_1}$, is admissible in U if and only if the following conditions hold:

- $\alpha_1 \geq a_2 - 1$;
- $x_3^{\alpha_3} \cdots x_n^{\alpha_n} \geq_{\mathrm{opdlex}} x_3^{e+1-b_3}$ or $x_3^{\alpha_3} \cdots x_n^{\alpha_n} = 1$.

For example, if $\delta_1 = x_1^2$, $\delta_2 = x_1 x_2^3$, $\delta_3 = x_1 x_2^2 x_3^3$ and $n = 3$ then the set of the admissible monomials in $U_6^{(1)} = \delta_1(K[x_1, x_2, x_3]_4)$ are

$$\{\delta_1 x_1^4\} \cup \{\delta_1 x_1^3 x_2, \delta_1 x_1^3 x_3\} \cup \{\delta_1 x_1^2 x_2^2, \delta_1 x_1^2 x_2 x_3\} \cup \{\delta_1 x_1 x_2^3, \delta_1 x_1 x_2^2 x_3\}.$$

Example 5.7. Let $U = x_1^2 S^{(1)} \biguplus x_1 x_2^3 S^{(2)}$. Suppose $c = \binom{n+2}{2} + 2$. Then

$$\max_{>_{\mathrm{dlex}}} \big\{ f \in U^{(1)} : \ f \text{ is admissible and } \#\{h \in U : h \leq_{\mathrm{dlex}} f\} \leq c \big\} = \delta_1 x_1^2.$$

Indeed,

$$\#\{h \in U : h \leq_{\mathrm{dlex}} \delta_1 x_1^2\} = \#\delta_1 S_{\leq 2}^{(1)} \biguplus \{\delta_2\} = \binom{n+2}{2} + 1$$

and

$$\begin{aligned} \#\{h \in U : h \leq_{\mathrm{dlex}} \delta_1 x_1 x_2^2\} &= \#\big(\delta_1 S_{\leq 3}^{(1)} \setminus \delta_1\{x_1^3, x_1^2 x_2, \ldots, x_1^2 x_n\}\big) \biguplus \delta_2 S_{\leq 2}^{(2)} \\ &= \binom{n+3}{3} > c. \end{aligned}$$

By Example 5.5, the lex-smallest admissible monomial in $U_5^{(1)}$ is $\delta_1 x_1 x_2^2$. Thus the extremal set $L \subset U$ with $\#L = c$ is

$$L = \delta_1 S_{\leq 2}^{(1)} \biguplus \delta_2\{1, x_n\}.$$

Example 5.8. In general, it is not easy to understand the shape of extremal sets, but in some special cases they are simple.

If $b_1 = b_2 = \cdots = b_t$ then any monomial in U is admissible. Thus any extremal set M in U is of the form

$$M = \{h \in U : h \leq_{\mathrm{dlex}} f\}$$

for some $f \in U$.

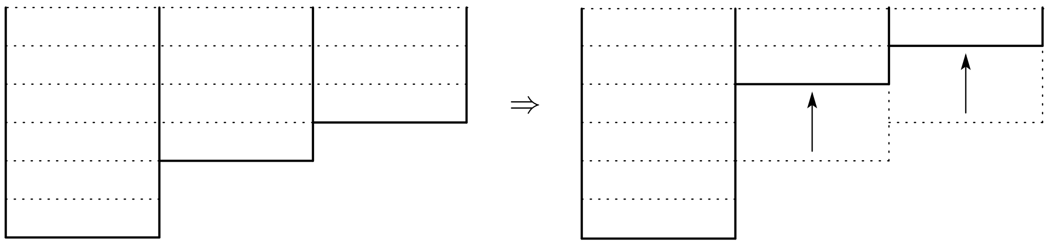
If $b_2 > e$ then the only admissible monomial in $U_e^{(1)}$ is $\delta_1 x_1^{e-b_1}$. Thus if $b_1 \ll b_2 \ll \cdots \ll b_n$ (for example, if $b_{i+1} - b_i > c$ for all i), then any extremal set M in U with $\#M = c$ is of the form

$$M = \delta_1 S_{\leq e_1}^{(1)} \biguplus \delta_2 S_{\leq e_2}^{(2)} \biguplus \cdots \biguplus \delta_{t-1} S_{\leq e_{t-1}}^{(t-1)} \biguplus N,$$

where $N \subset \delta_t S^{(t)}$ and $\# S_{\leq e_{i+1}}^{(i+1)} \biguplus \cdots \biguplus S_{\leq e_{t-1}}^{(t-1)} \biguplus N < \# S_{e_i+1}^{(i)}$ for $i = 1, 2, \ldots, t-1$.

In the rest of this section, we study properties of extremal sets. Suppose $t \geq 3$. For an integer $k \geq -a_3$, we write $U^{(i)}[-k] = (x_3^k \delta_i) S^{(i)}$. In the picture, $U^{(i)}[-k]$ is the picture obtained from that of $U^{(i)}$ by moving the blocks k steps above. In particular, for any integer $k \geq -a_3$, $U' = U^{(2)} \biguplus (\biguplus_{i=3}^t U^{(i)}[-k])$ is a universal lex ideal. (See Fig. 9.)

Figure 9



$U^{(\geq 2)}$ $\qquad\qquad$ $U' = U^{(2)} \biguplus (\biguplus_{i=3}^{t} U^{(i)}[-k])$

Lemma 5.9. *Suppose $t \geq 3$. Let $f \in U_e^{(1)}$, $d = \deg \rho(f)$ and $k \geq -a_3$ with $e-d+k \geq 0$. Then f is admissible over U if and only if the following conditions hold:*

- $\deg \rho(f) \leq e+1$ *or* $\rho(f) = \delta_2$*;*
- $\rho(f) x_2^{e-d+k}$ *is admissible in* $U' = U^{(2)} \biguplus (\biguplus_{i=3}^{t} U^{(i)}[-k])$*.*

Proof. Let ϕ be the moving map of U', $\delta_i' = x_3^k \delta_i$ and $\rho^i(f) = \delta_{i+1} u_{i+1}$ for $i = 2, \dots, t-1$. Then $\phi^i(\rho(f) x_2^{e-d+k}) = \delta_{i+2}' u_{i+2}$ for $i = 1, 2, \dots, t-2$. Thus $\deg \rho^i(f) \leq e+1$ if and only if $\deg \phi^{i-1}(\rho(f) x_2^{e-d+k}) \leq e+1+k$ for $i \geq 2$. Also, $\rho^{t-1}(f) \geq_{\mathrm{opdlex}} \delta_t x_t^{e+1-b_t}$ if and only if $\phi^{t-2}(\rho(f) x_2^{e+d+k}) \geq_{\mathrm{opdlex}} \delta_t' x_t^{e+1-b_t}$. Since $\deg \rho(f) x_2^{e-d+k} = e+k$, the above facts prove the statement. $\square$

By the definition of the maximal condition, the following facts are straightforward.

Lemma 5.10. *Let $M \subset U$ be an extremal set.*

(i) *If $\#M \geq \#U_{\leq e}$ then $M \supset U_{\leq e}$.*
(ii) *If $\#M \geq \#U_{\leq e-1}^{(1)} \biguplus U_{\leq e}^{(\geq 2)}$ then $M \supset U_{\leq e-1}^{(1)} \biguplus U_{\leq e}^{(\geq 2)}$.*

Proof. Since M is extremal, there exists an $f \in U^{(1)}$ such that

$$M^{(1)} = \{h \in U^{(1)} : h \leq_{\mathrm{dlex}} f\}.$$

(i) Since $\delta_1 x_1^{e-b_1}$ is admissible and $\{h \in U : h \leq_{\mathrm{dlex}} \delta_1 x_1^{e-b_1}\} = U_{\leq e}$, $f \geq_{\mathrm{dlex}} \delta_1 x_1^{e-b_1}$. Then $M^{(1)} \supset \{h \in U^{(1)} : h \leq_{\mathrm{dlex}} \delta_1 x_1^{e-b_1}\} = U_{\leq e}^{(1)}$. Also, $\#M^{(\geq 2)} \supset \#\{h \in U^{(\geq 2)} : h \leq_{\mathrm{dlex}} f\} \supset U_{\leq e}^{(2)}$ by the definition of the maximal condition. Then the statement follows by induction on t.

(ii). It is clear that $M \supset U_{\leq e-1}$ by (i). If $\deg f \geq e$ then

$$\#M \geq \#\{h \in U : h \leq_{\mathrm{dlex}} f\} = \#M^{(1)} \biguplus U_{\leq e}^{(\geq 2)}.$$

Then $\#M^{(\geq 2)} \geq \#U_{\leq e}^{(\geq 2)}$ and $M^{(\geq 2)} \supset U_{\leq e}^{(\geq 2)}$ by (i) as desired. If $\deg f < e$ then $M^{(1)} = U_{\leq e-1}^{(1)}$ and $\#M^{(\geq 2)} \geq \#U_{\leq e}^{(\geq 2)}$ by the assumption. Hence $M^{(\geq 2)} \supset U_{\leq e}^{(\geq 2)}$ by (i). $\square$

Corollary 5.11. *Extremal sets are ladder sets.*

Proof. If $M \subset U$ is extremal then $M^{(i)}$ is super rev-lex for all i by the maximal condition. It is enough to prove that if $M_e^{(1)} \neq \emptyset$ then $M \supset U_{\leq e}^{(\geq 2)}$. If $M_e^{(1)} \neq \emptyset$ then there exists an admissible monomial $f \in U_e^{(1)}$ such that

$$\# M \geq \#\{h \in U : h \leq_{\text{dlex}} f\} \geq \# U_{\leq e-1}^{(1)} \biguplus U_{\leq e}^{(\geq 2)}.$$

Then the statement follows from Lemma 5.10. □

Lemma 5.12. *Suppose $t \geq 2$. Let $M \subset U$ be an extremal set.*

(i) *If $a_2 > 0$ then $M_e^{(1)} \neq 0$ if and only if $\# M \geq \# U_{\leq e}^{(1)}$.*

(ii) *If $a_2 = 0$ and $M_e^{(1)} \neq 0$ then $\# M > \# U_{\leq e}^{(1)}$.*

Proof. Let $f \in U_e^{(1)}$ be the lex-smallest admissible monomial in $U_e^{(1)}$ over U.

(i) It suffices to prove that

$$\#\{h \in U : h \leq_{\text{dlex}} f\} = \# U_{\leq e}^{(1)}. \tag{2}$$

If $f = \delta_1 x_1^{e-b_1}$ then $f' = \delta_1 x_1^{e-b_1-1} x_2$ is not admissible. By the definition of the admissibility, one has $\deg \rho(f') = \deg \delta_2 x_2 > e+1$ and $b_2 > e$. In this case we have $\{h \in U : h \leq_{\text{dlex}} f\} = U_{\leq e}^{(1)}$.

Suppose $f \neq \delta_1 x_1^{e-b_1}$. We prove (2) by using induction on t. Suppose $t = 2$. Then $f = \delta_1 x_1^{a_2-1} x_2^{e+1-b_2}$, and

$$\{h \in U : h \leq_{\text{dlex}} f\} = U_{\leq e-1}^{(1)} \biguplus [f, \delta_1 x_n^{e-b_1}] \biguplus U_{\leq e}^{(2)}.$$

Since $\rho([f, \delta_1 x_n^{e-b_1}]) = \biguplus_{j=e+1}^{e+a_2} U_j^{(2)}$, we have

$$\#\{h \in U : h \leq_{\text{dlex}} f\} = \# U_{\leq e-1}^{(1)} + \# U_{\leq e+a_2}^{(2)} = \# U_{\leq e}^{(1)}$$

where we use $\rho(U_e^{(1)}) = U_{\leq e+a_2}^{(2)}$ for the last equality.

Suppose $t \geq 3$. Since $\rho(f) \neq \delta_2$, we have $\deg \rho(f) = e+1$. Indeed, by Lemma 5.9, $\deg \rho(f) \leq e+1$. On the other hand, since $\delta_1 x_2^{a_2-1} x_2^{e+1-b_2}$ is admissible over U, $f \leq_{\text{lex}} \delta_1 x_1^{a_2-1} x_2^{e+1-b_2}$. Thus $\deg \rho(f) \geq \deg \rho(\delta_1 x_1^{a_2-1} x_2^{e+1-b_2}) = e+1$.

Consider $U' = U^{(2)} \biguplus_{i=3}^{t} U^{(i)}[-1]$. By Lemma 5.9 (consider the case when $d = e+1$ and $k = 1$), $\rho(f)$ is the lex-smallest admissible monomial in $U_{e+1}^{(2)}$ over U'. Then

$$\begin{aligned} \#\big[\rho(f), \delta_2 x_n^{e+1-b_2}\big] \biguplus U_{\leq e}^{(\geq 2)} &= \#[\rho(f), \delta_2 x_n^{e+1-b_2}] \biguplus U_{\leq e}^{(2)} \biguplus U'^{(\geq 3)}_{\leq e+1} \\ &= \#\big\{h \in U' : h \leq_{\text{dlex}} \rho(f)\big\} \\ &= \# U_{\leq e+1}^{(2)} \end{aligned} \tag{3}$$

where the last equation follows from the induction hypothesis. On the other hand

$$\{h \in U : h \leq_{\text{dlex}} f\} = [f, \delta_1 x_n^{e-b_1}] \biguplus U_{\leq e-1}^{(1)} \biguplus U_{\leq e}^{(\geq 2)} \tag{4}$$

and

$$\rho\big([f, \delta_1 x_n^{e-b_1}]\big) = \big[\rho(f), \delta_2 x_n^{e+1-b_2}\big] \biguplus \left(\biguplus_{j=e+2}^{e+a_2} U_j^{(2)} \right). \tag{5}$$

(3), (4) and (5) show

$$\#\{h \in U : h \leq_{\mathrm{dlex}} f\} = \#U^{(1)}_{\leq e-1} \biguplus U^{(2)}_{\leq e+a_2} = \#U^{(1)}_{\leq e-1} \biguplus U^{(1)}_{e} = \#U^{(1)}_{\leq e}$$

where the second equality follows since $\rho(U^{(1)}_e) = U^{(2)}_{\leq e+a_2}$.

(ii) It suffices to prove that

$$\{h \in U : h \leq_{\mathrm{dlex}} f\} > \#U^{(1)}_{\leq e}.$$

Since $a_2 = 0$, $\#U^{(2)}_{\leq e} = \#U^{(1)}_e$. Then we have

$$\#\{h \in U : h \leq_{\mathrm{dlex}} f\} > \#U^{(1)}_{\leq e-1} \biguplus U^{(2)}_{\leq e} = \#U^{(1)}_{\leq e-1} \biguplus U^{(1)}_{e} = U^{(1)}_{\leq e},$$

as desired. $\square$

Corollary 5.13. *Suppose $t \geq 2$. Let $B \subset U^{(1)}_e$ be the rev-lex set and $N \subset U^{(\geq 2)}$ a ladder set with $\#N \geq \#U^{(\geq 2)}_{\leq e}$. Let $Y \subset U$ be the extremal set with $\#Y = \#U^{(1)}_{\leq e-1} \uplus B \uplus N$. If $\#B \uplus N < \#U^{(1)}_e$ then*

$$Y = U^{(1)}_{\leq e-1} \biguplus Y^{(\geq 2)}.$$

Proof. Since $\#Y \geq \#U_{\leq e-1}$, we have $Y \supset U_{\leq e-1}$ by Lemma 5.10. On the other hand, since $\#Y = \#U^{(1)}_{\leq e-1} \uplus B \uplus N < \#U^{(1)}_{\leq e}$ by the assumption, we have $Y^{(1)}_e = \emptyset$ by Lemma 5.12. Hence $Y^{(1)} = U^{(1)}_{\leq e-1}$. $\square$

For monomials $f >_{\mathrm{lex}} g \in U^{(i)}_j$, let $[f, g) = [f, g] \setminus \{g\}$.

Lemma 5.14. *Let $f \in U^{(1)}_e$ be the lex-smallest admissible monomial in $U^{(1)}_e$ over U and $g >_{\mathrm{lex}} h \in U^{(1)}_e$ admissible monomials over U such that there are no admissible monomials in $[g, h]$ except for g and h. Then $\#[g, h) \leq \#[f, \delta_1 x_n^{e-b_1}]$.*

Proof. If $t = 1$ then all monomials are admissible over U. If $t = 2$ then any monomial $w \in U^{(1)}_e$ with $w >_{\mathrm{lex}} f$ is admissible over U. Thus the statement is clear if $t \leq 2$.

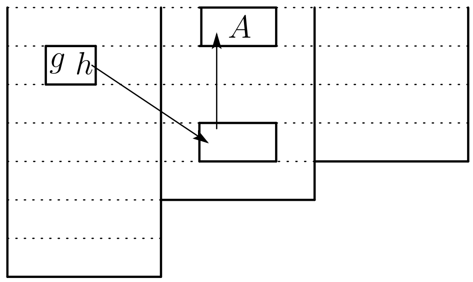
Suppose $t \geq 3$. Since $g \neq h$ we have $f \neq \delta_1 x_1^{e-b_1}$. By the definition of the admissibility, we have $\deg(\rho(f)) = e$ if $a_2 = 0$ and $\deg(\rho(f)) = e + 1$ if $a_2 > 0$. We consider the case when $a_2 > 0$ (the proof for the case when $a_2 = 0$ is similar).

Consider $U' = U^{(2)} \uplus (\biguplus_{i=3}^{t} U^{(i)}[-1])$. Since any monomial $w \in U^{(1)}_e$ such that $\rho(w) = \delta_2 x_2^k$ with $k \leq e + 1 - b_2$ is admissible over U, we have $\rho([g, h)) \subset S_d$ for some $d \leq e + 1$. Let

$$A = x_2^{e+1-d} \rho\big([g, h)\big) = \big[x_2^{e+1-d}\rho(g), x_2^{e+1-d}\rho(h)\big) \subset U^{(2)}_{e+1}.$$

(See Fig. 10.)

Figure 10



Let $w \in A$. Then $w = x_2^{e+1-d}\rho(w')$ for some $w' \in [g, h)$. Lemma 5.9 says that w is admissible over U' if and only if w' is admissible over U. Hence A contains no admissible monomial over U' except for $x_2^{e+1-d}\rho(g)$. By Lemma 5.9, $\rho(f) \in U_{e+1}^{(2)}$ is the lex-smallest admissible monomial in $U_{e+1}^{(2)}$ over U'. Then, by the induction hypothesis,

$$\#A \leq \#[\rho(f), \delta_2 x_n^{e-b_2}] = \#\rho\big([f, \delta_1 x_n^{e-b_1}]\big) \cap U_{e+1}^{(2)} \leq \#[f, \delta_1 x_n^{e-b_1}].$$

Then the statement follows since $\#[g, h) = \#\rho([g, h)) = \#A$. $\square$

Lemma 5.15. *Let $M \subset U$ be an extremal set, $e = \min\{k : \delta_1 x_1^{k-b_1} \notin M\}$ and $H = U_e \setminus M_e$. Let $f \in U_e^{(1)}$ be the lex-smallest admissible monomial in $U_e^{(1)}$ over U. Then*

(i) $\#U_{\leq e} + \#[f, \delta_1 x_n^{e-b_1}] \leq \#U_{\leq e+1}^{(1)}$.

(ii) $\#M + \#H < \#U_{\leq e+1}^{(1)}$.

Proof. We use induction on t. If $t = 1$ then then the statements are obvious. Suppose $t > 1$.

(i) If $a_2 > 0$ then by Lemma 5.12

$$\#U_{\leq e} + \#[f, \delta_1 x_n^{e-b_1}] = \#\{h \in U : h \leq_{\mathrm{dlex}} f\} + \#U_e^{(1)} = \#U_{\leq e}^{(1)} + \#U_e^{(1)} < \#U_{\leq e+1}^{(1)}$$

as desired. Suppose $a_2 = 0$. Then

$$\rho\big([f, \delta_1 x_n^{e-b_1}]\big) = [\rho(f), \delta_2 x_n^{e-b_2}] \subset U_e^{(2)}$$

and $\rho(f)$ is the lex-smallest admissible monomial in $U_e^{(2)}$ over $U^{(\geq 2)}$ by Lemma 5.9. Then by the induction hypothesis

$$\begin{aligned} \#U_{\leq e} + \#[f, \delta_1 x_n^{e-b_1}] &= \#U_{\leq e}^{(1)} + \big(\#U_{\leq e}^{(\geq 2)} + \#[\rho(f), \delta_2 x_n^{e-b_2}]\big) \\ &\leq \#U_{\leq e}^{(1)} + \#U_{\leq e+1}^{(2)} \\ &= \#U_{\leq e+1}^{(1)} \end{aligned}$$

as desired.

(ii) Suppose $M_e^{(2)} \neq U_e^{(2)}$. Then $M_e^{(1)} = \emptyset$. Since $M^{(\geq 2)}$ is extremal over $U^{(\geq 2)}$, by the induction hypothesis

$$\#M + \#H = \#U_{\leq e-1}^{(1)} \biguplus M^{(\geq 2)} + \#U_e^{(1)} \biguplus H^{(\geq 2)} < \#U_{\leq e}^{(1)} + \#U_{\leq e+1}^{(2)} \leq \#U_{\leq e+1}^{(1)},$$

where we use $\#U^{(1)}_{e+1} = \#U^{(2)}_{\leq e+1+a_2} \geq \#U^{(2)}_{\leq e+1}$ for the last inequality.

Suppose $M^{(2)}_e = U^{(2)}_e$. Let $g = \max_{>_{\mathrm{dlex}}} M^{(1)}$ and let

$$\mu = \min_{>_{\mathrm{dlex}}}\{h \in U^{(1)}_{\leq e} : h \text{ is admissible over } U \text{ and } h >_{\mathrm{dlex}} g\}.$$

Then $[\mu, g) \subset U^{(1)}_e$ since $g \geq_{\mathrm{dlex}} \delta_1 x_1^{e-b_1-1}$. Since M is extremal,

$$\#M < \#\{h \in U : h \leq_{\mathrm{dlex}} \mu\}.$$

Since $M^{(1)} = \{h \in U^{(1)} : h \leq_{\mathrm{dlex}} g\}$, $H = [\delta_1 x_1^{e-b_1}, g)$. Thus

$$\begin{aligned} \#M + \#H &< \#\{h \in U : h \leq_{\mathrm{dlex}} \mu\} + \#[\delta_1 x_1^{e-b_1}, g) \\ &= \#U_{\leq e} + \#[\mu, g) \\ &\leq \#U_{\leq e} + \#[f, \delta_1 x_n^{e-b_1}], \end{aligned}$$

where the last inequality follows from Lemma 5.14. Then the desired inequality follows from (i). $\square$

6. Proof of the main theorem

Let U be the universal lex ideal as in Section 5. The aim of this section is to prove the next result, which proves Proposition 3.9.

Theorem 6.1. *Let $M \subset U$ be a ladder set and $L \subset U$ the extremal set with $\#L = \#M$. Then $m(L) \succeq m(M)$.*

The proof of the above theorem is long. We prove it in subsections 6.1, 6.2 and 6.3 by case analysis.

In the rest of this section, we fix a ladder set $M \subset U$.

6.1. Preliminary of the proof.

For two subsets $A, B \subset U$, we define

$$A \gg B \Leftrightarrow \#A = \#B \text{ and } m(A) \succeq m(B).$$

Let $X \subset U^{(1)}$ be the super rev-lex set with $\#X = \#M^{(1)}$. Then $\{k : M^{(1)}_k \neq \emptyset\} \supset \{k : X_k \neq \emptyset\}$. Thus $X \cup M^{(\geq 2)}$ is also a ladder set in U. Since $X \gg M^{(1)}$ by Corollary 4.6, we have

Lemma 6.2. *There exists a ladder set $N \subset U$ such that $N^{(1)}$ is super rev-lex and $N \gg M$.*

Thus in the rest of this section, we assume that $M^{(1)}$ is super rev-lex. Let

$$e = \min\{k : \delta_1 x_1^{k-b_1} \notin M\}$$

and

$$f = \max_{>_{\mathrm{dlex}}}\{g \in U^{(1)}_{\leq e} : g \text{ is admissible over } U \text{ and } \#\{h \in U : h \leq_{\mathrm{dlex}} g\} \leq \#M\}.$$

Since $\delta_1 x_1^{e-b_1-1}$ is admissible over U, we have $f = \delta_1 x_1^{e-b_1-1}$ or $\deg f = e$. We will prove

Proposition 6.3. *With the same notation as above, there exists a ladder set N such that $N \gg M$ and*

$$N^{(1)} = \{h \in U^{(1)} : h \leq_{\mathrm{dlex}} f\}.$$

The above proposition proves Theorem 6.1. Indeed, by applying the above proposition repeatedly, one obtains a set N which satisfies the maximal condition and $N \gg M$. Then apply the induction on t. Also if $t = 1$ then Proposition 6.3 follows from Corollary 4.6. In the rest of this section, we assume that $t > 1$ and that the statement is true for universal lex ideals generated by at most $t-1$ monomials, and prove the proposition for U. By the above argument, we may assume that Theorem 6.1 is also true for universal lex ideals generated by at most $t-1$ monomials.

Lemma 6.4. *There exists a ladder set $N \subset U$ with $N \gg M$ and $\min\{k : \delta_1 x_1^{k-b_1} \notin N^{(1)}\} = e$ satisfying the following conditions*

(A1) *$N^{(1)}$ is super rev-lex and $N^{(\geq 2)}$ is extremal in $U^{(\geq 2)}$.*
(A2) *$\rho(N_e^{(1)}) \cup N^{(2)} \supset U^{(2)}_{\leq e+a_2}$ or $\rho(N_e^{(1)}) \cap N^{(2)} = \emptyset$.*
(A3) *If $t = 2$ and $\rho(N_e^{(1)}) \cap N^{(2)} = \emptyset$ then $N_e^{(1)} = \emptyset$. If $t \geq 3$ and $\rho(N_e^{(1)}) \cap N^{(2)} = \emptyset$ then $N_e^{(1)} = \emptyset$ or there exists a $d \geq e$ such that $N^{(2)} = U^{(2)}_{\leq d}$ and $N^{(3)}_{d+1} \neq U^{(3)}_{d+1}$.*

Proof. Let $F = M_e^{(1)}$. Then $M = (U^{(1)}_{\leq e-1} \biguplus F) \biguplus M^{(2)} \biguplus M^{(\geq 3)}$ since $M^{(1)}$ is super rev-lex.

Step 1. We first prove that there exits N satisfying $(A1)$. Let X be the extremal set in $U^{(\geq 2)}$ with $\# X = \# M^{(\geq 2)}$. Let

$$N = M^{(1)} \biguplus X = U^{(1)}_{\leq e-1} \biguplus F \biguplus X.$$

Since we assume that Theorem 6.1 is true for $U^{(\geq 2)}$, $N \gg M$. What we must prove is that N is a ladder set. Since $M^{(\geq 2)} \supset U^{(\geq 2)}_{\leq e-1}$, $\# X = \# M^{(\geq 2)} \geq \# U^{(\geq 2)}_{\leq e-1}$. Then Lemma 5.12 says $X \supset U^{(\geq 2)}_{\leq e-1}$, which shows that N is a ladder set if $F = \emptyset$. If $F \neq \emptyset$ then $M^{(\geq 2)} \supset U^{(\geq 2)}_{\leq e}$ by the definition of ladder sets, and $X \supset U^{(\geq 2)}_{\leq e}$ by Lemma 5.12. Hence N is a ladder set.

Step 2. We prove that if M satisfies $(A1)$ but does not satisfy either $(A2)$ or $(A3)$ then there exists an N satisfying (A2) and (A3) such that $\# N^{(1)}$ is strictly smaller than $\# M^{(1)}$. We may assume $\rho(F) \cup M^{(2)} \not\supset U^{(2)}_{\leq e+a_2}$. Let

$$a = \min\{k : M_k^{(2)} \neq U_k^{(2)}\},$$

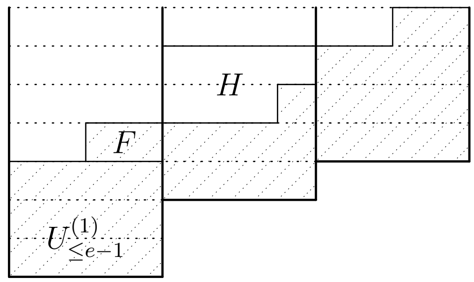
$$b = \max\{k : k \leq e + a_2,\ \rho(F)_k \neq U_k^{(2)}\}$$

and

$$d = \max\{k : M_k^{(3)} = U_k^{(3)}\}$$

where $d = \infty$ if $n = 2$. Let $H = U^{(2)}_{\leq d} \setminus M^{(2)}$. (See Fig. 11.)

Figure 11



M

Since $\rho(F)$ is an upper rev-lex set of degree $e+a_2$, $\rho(F)=\rho(F)_b \biguplus (\biguplus_{j=b+1}^{e+a_2} U_j^{(2)})$. If $H=\emptyset$ then $M^{(2)}=U^{(2)}_{\le d}$. Since $\rho(F)\cup M^{(2)} \not\supset U^{(2)}_{\le e+a_2}$, we have $b>d$ and $\rho(F)\cap M^{(2)}=\emptyset$, which says that M satisfies $(A2)$ and $(A3)$. Suppose $H\neq\emptyset$. Observe that for any super rev-lex set L with $U^{(2)}_{\le e}\subset L \subset U^{(2)}_{\le d}$, $M^{(1)} \biguplus L \biguplus M^{(\ge 3)}$ is a ladder set.

Case 1: Suppose $\#H \ge \#F$. Note that if $t=2$ then we always have $\#H\ge \#F$. Then $M^{(2)}$ is super rev-lex and $\rho(F)$ is an upper rev-lex set of degree $e+a_2$ with $\#M^{(2)}+\#\rho(F)\le \#U^{(2)}_{\le d}$. Let $R\subset U^{(2)}$ be the super rev-lex set in $U^{(2)}$ with $\#R=\#M^{(2)}+\#\rho(F)$. By Corollary 4.5,

$$m(R)\succeq m(M^{(2)})+m\big(\rho(F)\big)=m(M^{(2)})+m(F). \tag{6}$$

Also, since R is super rev-lex, $U^{(2)}_{\le e}\subset R\subset U^{(2)}_{\le d}$. Thus

$$N=U^{(1)}_{\le e-1}\biguplus R \biguplus M^{(\ge 3)}$$

is a ladder set. Then $N^{(1)}_e=\emptyset$ and $N\gg M$ by (6). Hence N satisfies (A2) and (A3).

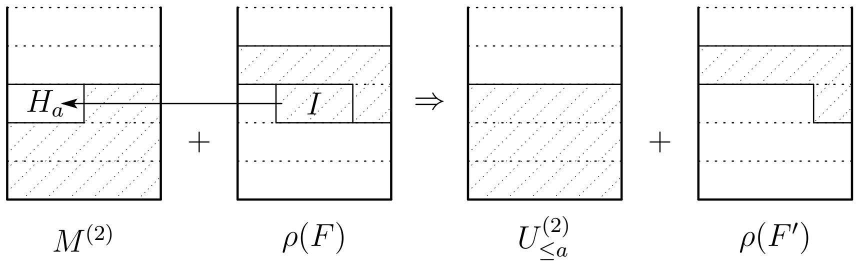
Case 2: Suppose $\#H<\#F$. Observe that $M^{(2)}\cup\rho(F)$ contains all monomials of degree k in $U^{(2)}$ for $k<a$ and $b<k\le e+a_2$. Since $M\cup\rho(F)\not\supset U^{(2)}_{\le e+a_2}$, we have $a\le b$.

Let $I\subset\rho(F)$ be the interval in $U^{(2)}$ such that $\#I=\#H_a$ and $\rho(F)\setminus I$ is an upper rev-lex set of degree $e+a_2$, and let $F'\subset F$ be the rev-lex set with $\rho(F')=\rho(F)\setminus I$. Since H_a is a lower lex set of degree a, by the interval lemma,

$$m\big(M^{(2)}\big)+m\big(\rho(F)\big)\ll m\left(H_a\biguplus M^{(2)}\right)+m\big(\rho(F)\setminus I\big)=m\big(U^{(2)}_{\le a}\big)+m\big(\rho(F')\big).$$

(See Fig. 12.)

Figure 12



If $\rho(F') \cup U^{(2)}_{\le a} \supset U^{(2)}_{\le e+a_2}$ then

$$N = \big(U^{(1)}_{\le e-1} \biguplus F'\big) \biguplus U^{(2)}_{\le a} \biguplus M^{(\ge 3)}$$

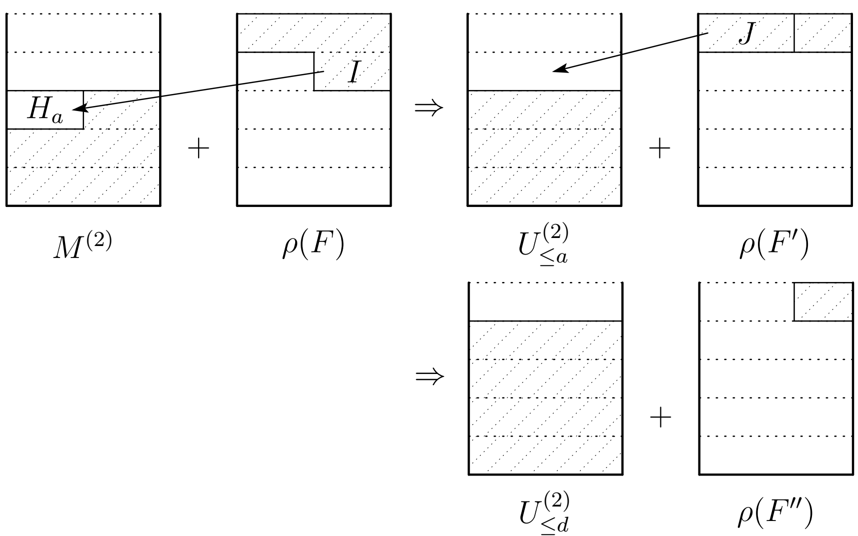
is a ladder set and satisfies $N \gg M$ and conditions (A2) and (A3) since $\rho(N^{(1)}_e) \cup N^{(2)} \supset U^{(2)}_{\le e+a_2}$.

Suppose $\rho(F') \cup U^{(2)}_{\le a} \not\supset U^{(2)}_{\le e+a_2}$. Then $\rho(F') \subset \biguplus_{j=a+1}^{e+a_2} U^{(2)}_j$. Since we assume $\#H < \#F$, $\#F' = \#F - \#H_a > \#(H \setminus H_a)$. Let $J \subset \rho(F')$ be the interval in $U^{(2)}$ such that $\#J = \#(H \setminus H_a)$ and $\rho(F') \setminus J$ is an upper rev-lex set of degree $e + a_2$, and let $F'' \subset F'$ be the rev-lex set satisfying $\rho(F'') = \rho(F') \setminus J$. Since $H \setminus H_a = \biguplus_{j=a+1}^{d} U^{(2)}_j$ is a lower lex set of degree $a+1$, by the interval lemma

$$m\big(U^{(2)}_{\le a}\big) + m\big(\rho(F')\big) \preceq m\big(M^{(2)} \biguplus H\big) + m\big(\rho(F'')\big) = m\big(U^{(2)}_{\le d}\big) + m\big(\rho(F'')\big).$$

(See Fig. 13.)

Figure 13



Then

$$N = \big(U^{(1)}_{\le e-1} \biguplus F''\big) \biguplus U^{(2)}_{\le d} \biguplus M^{(\ge 3)}$$

is a ladder set and satisfies $N \gg M$ and conditions (A2) and (A3).

Finally, since Step 1 does not change the first component $M^{(1)}$ and Step 2 decreases the first component, by applying Step 1 and 2 repeatedly, we obtain a set $N \subset U$ satisfying conditions (A1), (A2) and (A3). $\square$

Lemma 6.4 says that to prove Proposition 6.3 we may assume that M satisfies (A1), (A2) and (A3). Thus in the rest of this section we assume that M satisfies these conditions.

6.2. **Proof of Proposition 6.3 when $f \neq \delta_1 x_1^{e-b_1-1}$.**

In this subsection, we prove Proposition 6.3 when $f \neq \delta_1 x_1^{e-b_1-1}$. In this case we have $\deg f = e$. Let

$$f = \delta_1 x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

and $F = M_e^{(1)}$. Since $\delta_1 x_1^{e-b_1} \notin F$ by the choice of e, we have $m(F) = m(\rho(F))$. Also we have

$$M^{(\geq 2)} \supset U_{\leq e}^{(\geq 2)}.$$

Indeed, this is obvious when $F \neq \emptyset$ by the definition of ladder sets. If $F = \emptyset$ then

$$\#M^{(\geq 2)} = \#M - \#U_{\leq e-1}^{(1)} \geq \#\{h \in U : h \leq_{\mathrm{dlex}} f\} - \#U_{\leq e-1}^{(1)} \geq \#U_{\leq e}^{(2)},$$

and since $M^{(\geq 2)}$ is extremal we have $M^{(\geq 2)} \supset U_{\leq e}^{(\geq 2)}$ by Lemma 5.10. Let

$$\epsilon = \deg \rho(f) = \alpha_2 + \cdots + \alpha_n + b_2.$$

Case 1. Suppose $\rho(F) \subset \biguplus_{j=\epsilon}^{e+a_2} U_j^{(2)}$ and $\#F + \#M^{(2)} \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)} \leq \#U_{\leq e+a_2}^{(2)}$. Observe $M^{(2)} \supset \biguplus_{j=\epsilon}^{e} U_j^{(2)}$. Let P be the super rev-lex set with $\#P = \#M^{(2)} \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)}$, and let $Q \subset U^{(2)}$ be the super rev-lex set with $\#Q = \#F + \#M^{(2)} \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)}$. Since $\rho(F)$ is an upper rev-lex set of degree $e + a_2$ and $M^{(2)} \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)}$ is rev-lex, by Corollaries 4.5 and 4.6, we have

$$m(Q) \succeq m(P) + m\big(\rho(F)\big) \succeq m\Big(M^{(2)} \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)}\Big) + m(F). \tag{7}$$

(See the first two steps in Fig. 15.)

Observe that $Q \subset U_{\leq e+a_2}^{(2)}$ since $\#Q \leq \#U_{\leq e+a_2}^{(2)}$ by the assumption of Case 1. Let $U' = U^{(2)} \biguplus (\biguplus_{i=3}^{t} U^{(i)}[-a_2])$. Since $M^{(\geq 3)}[-a_2] \supset U_{\leq e}^{(\geq 3)}[-a_2] = {U'}_{\leq e+a_2}^{(\geq 3)}$,

$$Q \biguplus M^{(\geq 3)}[-a_2] \subset U'$$

is a ladder set in U'. (See the third step in Fig. 15.)

Let g be the largest admissible monomial in $U_{\leq e+a_2}^{(2)}$ over U' with respect to $>_{\mathrm{dlex}}$ satisfying

$$\#\{h \in U' : h \leq_{\mathrm{dlex}} g\} \leq \#Q \biguplus M^{(\geq 3)}.$$

By the induction hypothesis, there exists $Y \subset {U'}^{(\geq 3)}$ such that

$$X = \{h \in U^{(2)} : h \leq_{\mathrm{dlex}} g\} \biguplus Y \subset U'$$

is a ladder set in U' and

$$X \gg Q \biguplus M^{(\geq 3)}. \tag{8}$$

Let

$$d = e + a_2 - \epsilon.$$

We claim

Lemma 6.5. $g \geq_{\mathrm{lex}} x_2^d \rho(f)$.

Proof. To prove this, consider

$$L = \{h \in U : h \leq_{\mathrm{dlex}} f\}.$$

Then $\# M \geq \# L$ and $L^{(\geq 2)} = U^{(\geq 2)}_{\leq e}$. Let $F' = L^{(1)}_e = [f, \delta_1 x_n^{e-b_1}]$. Then $\rho(F') = [\rho(f), \delta_2 x_n^{\epsilon - b_2}] \biguplus (\biguplus_{j=\epsilon+1}^{e+a_2} U_j^{(2)})$. Also $\rho(F') \cap (L^{(2)} \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)}) = \emptyset$ and

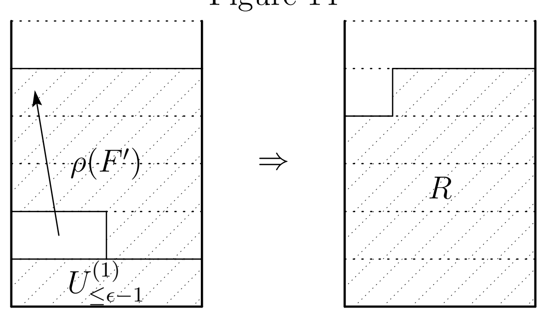
$$\begin{aligned} m\big(\rho(F') \biguplus \big(L^{(2)} \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)}\big)\big) &= m\big(U^{(2)}_{\leq e+a_2} \setminus \big[\delta_2 x_2^{\epsilon - b_2}, \rho(f)\big)\big) \\ &= m\big(U^{(2)}_{\leq e+a_2} \setminus \big[\delta_2 x_2^{e+a_2-b_2}, x_2^d \rho(f)\big)\big). \end{aligned}$$

Let

$$R = U^{(2)}_{\leq e+a_2} \setminus \big[\delta_2 x_2^{e+a_2-b_2}, x_2^d \rho(f)\big) = U^{(2)}_{\leq e+a_2-1} \biguplus \big[x_2^d \rho(f), \delta_2 x_n^{e+a_2-b_2}\big].$$

(See Fig. 14).

Figure 14



$U^{(1)}_{\leq \epsilon-1} \biguplus \rho(F')$ $\qquad$ R

Then $R \biguplus L^{(\geq 3)}[-a_2] \subset U'$ is a ladder set in U' and $x_2^d \rho(f)$ is admissible over U' by Lemma 5.9. On the other hand,

$$\# R \biguplus L^{(\geq 3)} = \# L - \# U^{(1)}_{\leq e-1} - \# \biguplus_{j=\epsilon}^{e} U_j^{(2)} \leq \# M - \# U^{(1)}_{\leq e-1} - \# \biguplus_{j=\epsilon}^{e} U_j^{(2)} = \# X.$$

Since $x_2^d \rho(f)$ is admissible over U' and since $R \biguplus L^{(\geq 3)}[-a_2] = \{h \in U' : h \leq_{\mathrm{dlex}} x_2^d \rho(f)\}$, by the choice of g, we have

$$g \geq_{\mathrm{lex}} x_2^d \rho(f)$$

as desired. $\square$

Let $H \subset U_e^{(1)}$ be the rev-lex set such that

$$\rho(H) = \biguplus_{j=\epsilon}^{e+a_2} U_j^{(2)} \setminus x_2^{-d}\big[\delta_2 x_2^{e+a_2-b_2}, g\big).$$

Then by Lemma 4.3

$$m(H) + m\big(U_{\leq \epsilon-1}^{(2)}\big) \succeq m\big(U_{\leq e+a_2}^{(2)} \setminus [\delta_2 x_2^{e+a_2-b_2}, g)\big) = m\big(X^{(2)}\big). \tag{9}$$

Let

$$N = \big(U_{\leq e-1}^{(1)} \biguplus H\big) \biguplus U_{\leq e}^{(2)} \biguplus Y[+a_2] \subset U.$$

Since $Y \supset {U'}_{\leq e+a_2}^{(\geq 3)}$, we have $Y[+a_2] \supset U_{\leq e}^{(\geq 3)}$. Thus N is a ladder set in U. We claim that N satisfies the desired conditions.

Let $\mu = \max_{>_{\mathrm{lex}}} H$. Then $x_2^d \rho(\mu) = g$. We claim that $\mu = f$. Since $g \geq_{\mathrm{lex}} x_2^d \rho(f)$, $\mu \geq_{\mathrm{lex}} f$. Since g is admissible over U', μ is admissible over U by Lemma 5.9. (If $t = 2$ then Lemma 5.9 is not applicable, however, if $t = 2$ then any monomial $h \in U_e^{(1)}$ with $h >_{\mathrm{lex}} f$ is admissible). However, since $\#N = \#M$ and $N \supset \{h \in U : h \leq_{\mathrm{dlex}} \mu\}$, by the choice of f, we have $f = \mu$.

It remains to prove $N \gg M$. This follows from (7), (8) and (9) as follows:

$$\begin{aligned}
M \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)} &= \big(U_{\leq e-1}^{(1)} \biguplus F\big) \biguplus \big(M^{(2)} \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)}\big) \biguplus M^{(\geq 3)} \\
&\ll U_{\leq e-1}^{(1)} \biguplus Q \biguplus M^{(\geq 3)} \\
&\ll U_{\leq e-1}^{(1)} \biguplus X \\
&\ll \big(U_{\leq e-1}^{(1)} \biguplus H\big) \biguplus U_{\leq \epsilon-1}^{(2)} \biguplus Y[+a_2] = N \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)}.
\end{aligned}$$

(See Fig. 15.)

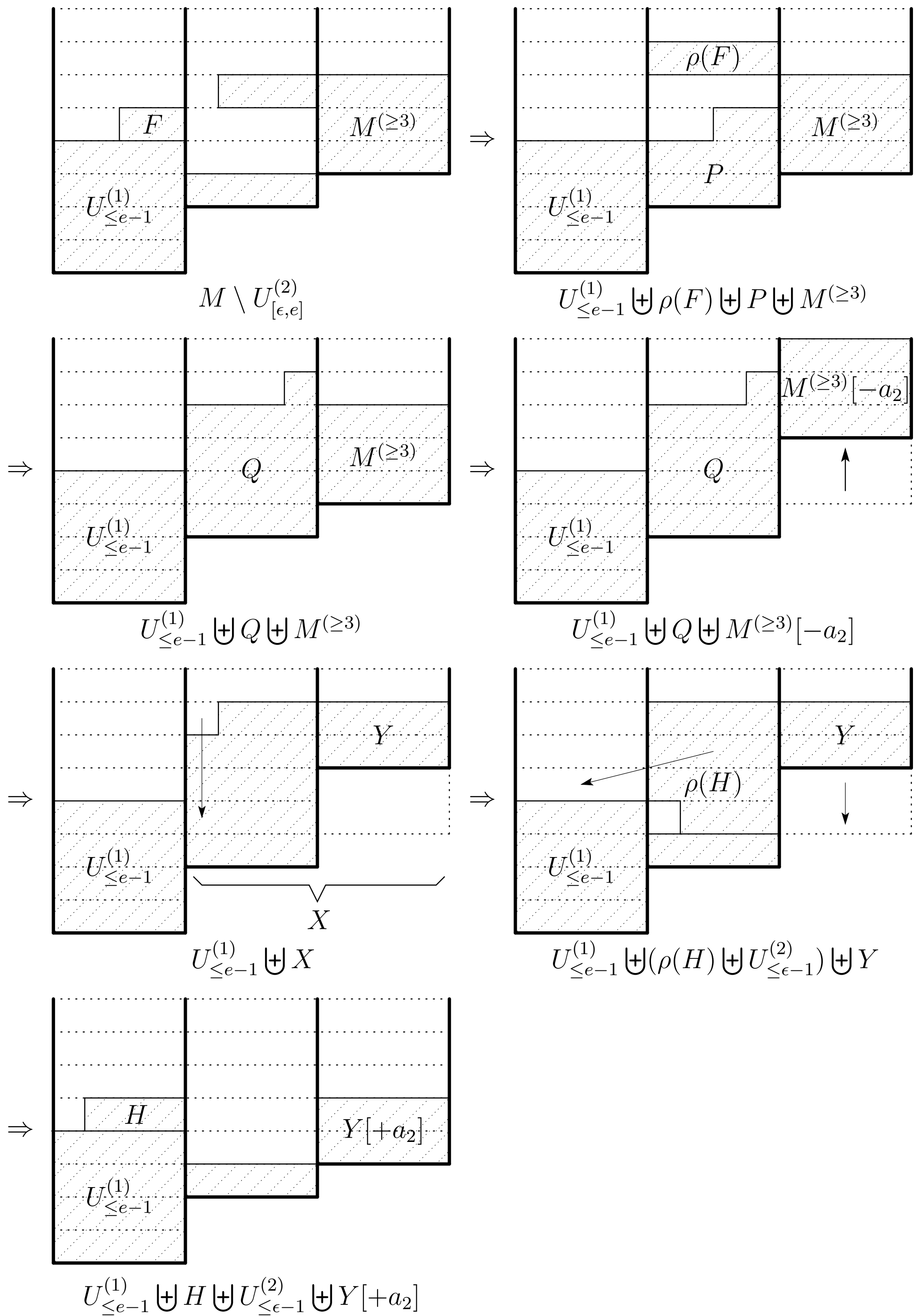
Case 2. Suppose $\rho(F) \subset \biguplus_{j=\epsilon}^{e+a_2} U_j^{(2)}$ and $\#F + \#M^{(2)} \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)} > \#U_{\leq e+a_2}^{(2)}$. We claim

Lemma 6.6. *$f = \delta_1 x_1^{\alpha_1} x_2^{\alpha_2}$, that is, $\alpha_3 = \cdots = \alpha_n = 0$.*

Proof. Suppose $f \neq \delta_1 x_1^{\alpha_1} x_2^{\alpha_2}$. Let $g = \delta_1 x_1^{\alpha_1} x_2^{\alpha_2+\alpha_3+\cdots+\alpha_n}$. Then $g >_{\mathrm{dlex}} f$ is admissible over U by the definition of the admissibility. Also,

$$\#M < \#\{h \in U : h \leq_{\mathrm{dlex}} g\} = \#\big(U_{\leq e-1}^{(1)} \biguplus [g, \delta_1 x_n^{e-b_1}]\big) \biguplus U_{\leq e}^{(2)} \biguplus U_{\leq e}^{(\geq 3)}.$$

Figure 15


F
$M^{(\geq 3)}$
$U^{(1)}_{\leq e-1}$
$M \setminus U^{(2)}_{[\epsilon,e]}$
$\Rightarrow$
$\rho(F)$
$M^{(\geq 3)}$
P
$U^{(1)}_{\leq e-1}$
$U^{(1)}_{\leq e-1} \biguplus \rho(F) \biguplus P \biguplus M^{(\geq 3)}$
$\Rightarrow$
Q
$M^{(\geq 3)}$
$U^{(1)}_{\leq e-1}$
$U^{(1)}_{\leq e-1} \biguplus Q \biguplus M^{(\geq 3)}$
$\Rightarrow$
$M^{(\geq 3)}[-a_2]$
Q
$U^{(1)}_{\leq e-1}$
$U^{(1)}_{\leq e-1} \biguplus Q \biguplus M^{(\geq 3)}[-a_2]$
$\Rightarrow$
Y
$U^{(1)}_{\leq e-1}$
X
$U^{(1)}_{\leq e-1} \biguplus X$
$\Rightarrow$
Y
$\rho(H)$
$U^{(1)}_{\leq e-1}$
$U^{(1)}_{\leq e-1} \biguplus (\rho(H) \biguplus U^{(2)}_{\leq \epsilon-1}) \biguplus Y$
$\Rightarrow$
H
$Y[+a_2]$
$U^{(1)}_{\leq e-1}$
$U^{(1)}_{\leq e-1} \biguplus H \biguplus U^{(2)}_{\leq \epsilon-1} \biguplus Y[+a_2]$

Since $\rho([g,\delta_1 x_n^{e-b_1}]) = \biguplus_{i=\epsilon}^{e+a_2} U_i^{(2)}$ and $M^{(\geq 3)} \supset U_{\leq e}^{(\geq 3)}$,

$$\begin{aligned} \#F + \#\big(M^{(2)} \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)}\big) &= \big(\#M - \#U_{\leq e-1}^{(1)} - \#M^{(\geq 3)}\big) - \#\biguplus_{j=\epsilon}^{e} U_j^{(2)} \\ &< \#[g,\delta_1 x_n^{e-b_1}] + \#U_{\leq e}^{(2)} - \#\biguplus_{j=\epsilon}^{e} U_j^{(2)} \\ &= \#U_{\leq e+a_2}^{(2)}, \end{aligned}$$

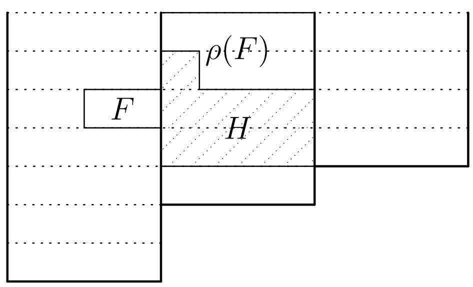
which contradicts the assumption of Case 2. Thus $f = \delta_1 x_1^{\alpha_1} x_2^{\alpha_2}$. $\square$

Note that the above lemma says $\rho(f) = \delta_2 x_2^{\epsilon-b_2}$. In particular, $\rho([f,\delta_1 x_n^{e-b_1}]) = \bigcup_{j=\epsilon}^{e+a_2} U_j^{(2)}$. Let

$$H = \biguplus_{j=\epsilon}^{e+a_2} U_j^{(2)} \setminus \rho(F).$$

(See Fig. 16).

Figure 16



H

Since $\rho(F)$ is an upper rev-lex set of degree $e + a_2$, H is a lower lex set of degree ϵ. Also, since $\#F + \#M^{(2)} > \#U_{\leq e+a_2}^{(2)}$, $\rho(F) \cup M^{(2)} \supset U_{\leq e+a_2}^{(\geq 2)}$ by (A2). Thus $M^{(2)} \supset H$.

Let R be the super rev-lex set in $U^{(2)}$ with $\#R = \#M^{(2)} \setminus H$. Since $M^{(2)} \setminus H$ is rev-lex, by Corollary 4.6 we have

$$R \gg M^{(2)} \setminus H. \tag{10}$$

Then since $\#R \leq \#M^{(2)}$,

$$R \biguplus M^{(\geq 3)} \subset U^{(\geq 2)}$$

is a ladder set. (See the third picture in Fig. 17.)

Let $Y \subset U^{(\geq 2)}$ be the extremal set in $U^{(\geq 2)}$ with $\#Y = \#R \biguplus M^{(\geq 3)}$. We claim that

$$N = \{h \in U^{(1)} : h \leq_{\text{dlex}} f\} \biguplus Y$$

satisfies the desired conditions. Indeed, since $\rho(F) \biguplus H = \biguplus_{j=\epsilon}^{e+a_2} U_j^{(2)} = \rho([f, \delta_2 x_n^{e-b_1}])$, by (10), we have

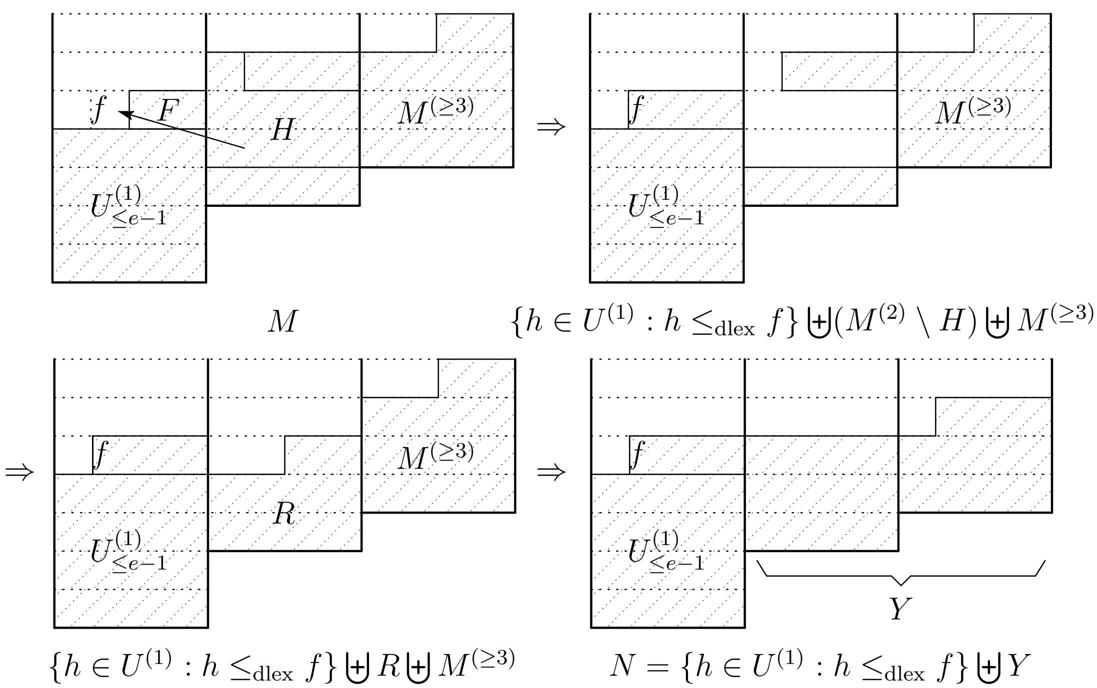
$$\begin{aligned} M &= \big(U_{\leq e-1}^{(1)} \biguplus F\big) \biguplus M^{(2)} \biguplus M^{(\geq 3)} \\ &= \big(U_{\leq e-1}^{(1)} \biguplus F \biguplus H\big) \biguplus \big(M^{(2)} \setminus H\big) \biguplus M^{(\geq 3)} \\ &\ll \big(U_{\leq e-1}^{(1)} \biguplus [f, \delta_1 x_n^{e-b_1}]\big) \biguplus R \biguplus M^{(\geq 3)} \\ &\ll \{h \in U^{(1)} : h \leq_{\mathrm{dlex}} f\} \biguplus Y = N. \end{aligned}$$

(See Fig. 17.) It remains to prove that N is a ladder set. Since

$$\#Y = \#M - \#\{h \in U^{(1)} : h \leq_{\mathrm{dlex}} f\} \geq \#U_{\leq e}^{(\geq 2)}$$

by the choice of f, we have $Y \supset U_{\leq e}^{(\geq 2)}$ by Lemma 5.10. This fact guarantees that N is a ladder set.

Figure 17



Case 3. Suppose $\rho(F) \not\subset \biguplus_{j=\epsilon}^{e+a_2} U_j^{(2)}$. Then $\rho(F)$ properly contains $\biguplus_{j=\epsilon}^{e+a_2} U_j^{(2)}$ since $\rho(F)$ is an upper rev-lex set of degree $e + a_2$. In particular, F properly contains $[f, \delta_1 x_n^{e-b_1}]$. We claim

Lemma 6.7. *$f = \delta_1 x_1^{\alpha_1} x_2^{\alpha_2}$ and $\alpha_2 \neq 0$.*

Proof. If $\alpha_k \neq 0$ for some $k \geq 3$ then $\delta_1 x_1^{\alpha_1} x_2^{\alpha_2+\cdots+\alpha_n} >_{\text{dlex}} f$ is admissible over U. Then by the choice of f, $F \subset [\delta_1 x_1^{\alpha_1} x_2^{\alpha_2+\cdots+\alpha_n}, \delta_1 x_n^{e-b_1}]$ and

$$\rho(F) \subset \rho\big([\delta_1 x_1^{\alpha_1} x_2^{\alpha_2+\cdots+\alpha_n}, \delta_1 x_n^{e-b_1}]\big) = \biguplus_{j=\epsilon}^{e+a_2} U_j^{(2)},$$

a contradiction. Also, if $\alpha_2 = 0$ then $\epsilon = \deg \rho(f) = 0$ which implies $\rho(F) \subset \rho(U_e^{(1)}) = U_{\leq e+a_2}^{(2)} = \biguplus_{j=\epsilon}^{e+a_2} U_j^{(2)}$, a contradiction. $\square$

Recall $\epsilon = \deg \rho(f)$. Thus $\alpha_2 = \epsilon - b_2$. Let

$$H = \{h \in F : h >_{\text{lex}} f\}$$

and

$$g = \max_{>_{\text{lex}}} H.$$

By the choice of f, H contains no admissible monomials over U. By Lemma 6.7, $\rho(F \setminus H) = \biguplus_{j=\epsilon}^{e+a_2} U_j^{(2)}$. Hence $H \neq \emptyset$ by the assumption of Case 3. Since $\delta_1 x_1^{\alpha_1+1} x_2^{\alpha_2-1}$ is admissible over U,

$$\rho(H) \subset \rho\big([\delta_1 x_1^{\alpha_1+1} x_2^{\alpha_2-1}, \delta_1 x_1^{\alpha_1} x_2^{\alpha_2})\big) = U_{\epsilon-1}^{(2)}$$

is rev-lex. Also, $\epsilon - 1 > b_2$ since $U_{b_2}^{(2)} = \{\delta_2\}$ and $H \neq \emptyset$.

If $t = 2$ then any monomial $h \in U_e^{(1)}$ with $h >_{\text{lex}} f$ is admissible, which implies $H = \emptyset$. Thus we may assume $t \geq 3$.

To prove the statement, it is enough to prove that there exists $Z \subset U^{(\geq 3)}$ such that

$$Z \gg H \biguplus M^{(\geq 3)}. \tag{11}$$

Indeed, if such a Z exists then $N = (M^{(1)} \setminus H) \uplus M^{(2)} \uplus Z$ satisfies the desired conditions. Recall that $\epsilon \leq e + 1$ by Definition 5.2.

(subcase 3-1) Suppose $a_3 \geq e - (\epsilon - 1)$.

Let $d = e - (\epsilon - 1)$. Then

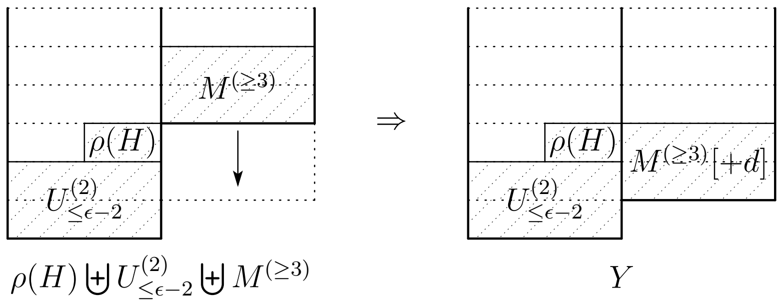
$$U' = U^{(2)} \biguplus \left(\biguplus_{i=3}^{t} U^{(i)}[+d] \right)$$

is universal lex. Recall $\rho(H) \subset U_{\epsilon-1}^{(2)}$. Let

$$Y = \rho(H) \biguplus U_{\leq \epsilon-2}^{(2)} \biguplus M^{(\geq 3)}[+d].$$

(See Fig. 18.) Then Y is a ladder set since $M^{(\geq 3)} \supset U_{\leq \epsilon-1+d}^{(\geq 3)} = U_{\leq e}^{(\geq 3)}$. Also, $U_{\leq \epsilon-2}^{(2)} \neq \emptyset$ since $\epsilon - 1 > b_2$.

Figure 18



Let $\mu \in U^{(2)}_{\leq \epsilon-1}$ be the largest admissible monomial in $U^{(2)}_{\leq \epsilon-1}$ over U' with respect to $>_{\mathrm{dlex}}$ satisfying $\#\{h \in U' : h \leq_{\mathrm{dlex}} \mu\} \leq \#Y$. Then since we assume that Proposition 6.3 is true for U', there exists $Z \subset U'^{(\geq 3)}$ such that

$$Y \ll \{h \in U^{(2)} : h \leq_{\mathrm{dlex}} \mu\} \biguplus Z.$$

To prove (11), it is enough to prove $\{h \in U^{(2)} : h \leq_{\mathrm{dlex}} \mu\} = U^{(2)}_{\leq \epsilon-2}$, in other words,

Lemma 6.8. $\mu = \delta_2 x_2^{\epsilon-2-b_2}$.

Proof. Recall that $U^{(2)}_{\leq \epsilon-2} \neq \emptyset$. It is enough to prove that $\deg \mu \neq \epsilon - 1$. Suppose contrary that $\deg \mu = \epsilon - 1$. Let $\mu' \in U^{(1)}_e$ be a monomial such that $\rho(\mu') = \mu$. Then μ' is admissible over U by Lemma 5.9. Also

$$\#Y - \#U^{(2)}_{\leq \epsilon-2} \geq \#[\mu, \delta_2 x_n^{\epsilon-1-b_2}] + \#U'^{(\geq 3)}_{\leq \epsilon-1} = \#[\mu, \delta_2 x_n^{\epsilon-1-b_2}] + \#U^{(\geq 3)}_{\leq e}.$$

Since $\#M^{(\geq 3)} + \#H = \#Y - \#U^{(2)}_{\leq \epsilon-2}$ and since $\rho([\mu', f)) = [\mu, \delta_2 x_n^{\epsilon-1-b_2}]$, we have

$$\begin{aligned} \#M &= \#(M \setminus H) \biguplus M^{(2)} \biguplus H \biguplus M^{(\geq 3)} \\ &\geq \#(M \setminus H) \biguplus U^{(2)}_{\leq e} \biguplus [\mu, \delta_2 x_n^{\epsilon-1-b_2}] \biguplus U^{(\geq 3)}_{\leq e} \\ &\geq \#[\mu', f) \biguplus (M \setminus H) \biguplus U^{(\geq 2)}_{\leq e} = \#\{h \in U : h \leq_{\mathrm{dlex}} \mu'\}, \end{aligned}$$

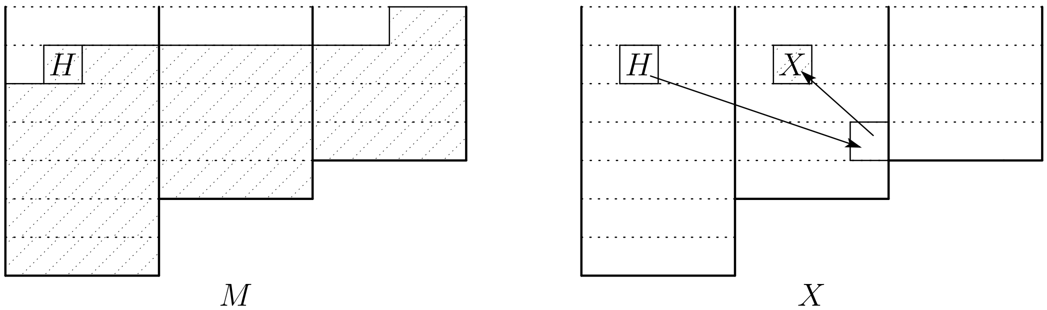
which contradicts the choice of f since $\mu' >_{\mathrm{lex}} g >_{\mathrm{lex}} f$ and μ' is admissible over U. $\square$

(subcase 3-2) Suppose $a_3 < e - (\epsilon - 1)$. We consider

$$X = x_2^{e-(\epsilon-1)} \rho(H).$$

(See Fig. 19.)

Figure 19



Let

$$Y = \{h \in U^{(2)} : h \leq_{\mathrm{dlex}} x_2^{e-(\epsilon-1)}\rho(g)\} \biguplus M^{(\geq 3)}$$

(see Fig. 20) and let

$$g' = \max_{>_{\mathrm{dlex}}} \big(Y^{(2)} \setminus X\big).$$

Since $e-(\epsilon-1) > a_3$, $e-(\epsilon-1) \geq 1$. Thus

$$g' = \delta_2 x_2^{e-(\epsilon-1)-1} x_3^{\epsilon-b_2}$$

and

$$Y^{(2)} = X \biguplus \{h \in U^{(2)} : h \leq_{\mathrm{dlex}} g'\}.$$

Since $a_3 < e-(\epsilon-1)$, $\deg \rho(\delta_2 x_2^{e-(\epsilon-1)-1} x_3^{\epsilon-b_2}) = \epsilon + a_3 \leq e$. Thus g' is admissible over $U^{(\geq 2)}$.

Let μ be the largest admissible monomial in $U^{(2)}_{\leq e}$ over $U^{(\geq 2)}$ with respect to $>_{\mathrm{dlex}}$ with $\#\{h \in U^{(\geq 2)} : h \leq_{\mathrm{dlex}} \mu\} \leq \#Y$. Since Lemma 5.9 says that X contains no admissible monomials over $U^{(\geq 2)}$,

$$\mu \geq_{\mathrm{dlex}} g' \text{ and } \mu \notin X.$$

Since we assume that Proposition 6.3 is true for $U^{(\geq 2)}$, there exists $Z \subset U^{(\geq 3)}$ such that

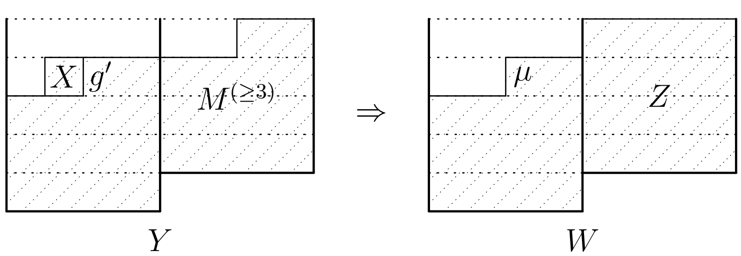
$$W = \{h \in U^{(2)} : h \leq_{\mathrm{dlex}} \mu\} \biguplus Z$$

is a ladder set and

$$W \gg Y$$

(See Fig. 20.)

Figure 20



We claim

Lemma 6.9. *$\mu = g'$.*

Proof. Suppose contrary that $\mu \neq g'$. Then $\mu >_{\mathrm{dlex}} g'$ and

$$W = \big[\mu, x_2^{e-(\epsilon-1)}\rho(g)\big) \biguplus Y^{(2)} \biguplus Z.$$

Then there exists $\mu' \in U_e^{(1)}$ such that

$$x_2^{e-(\epsilon-1)}\rho(\mu') = \mu.$$

By Lemma 5.9, μ' is admissible over U and $\mu' >_{\mathrm{lex}} g >_{\mathrm{lex}} f$. Observe that

$$\#M^{(\geq 3)} + \#H = \#Z \biguplus \big[\mu, x_2^{e-(\epsilon-1)}\rho(g)\big) \biguplus X = \#Z + \#[\mu', f)$$

by the construction of Y and Z. Since $Z \supset U_{\leq e}^{(\geq 3)}$,

$$\begin{aligned} \#M &\geq \#\big(M^{(1)} \setminus H\big) \biguplus H \biguplus U_{\leq e}^{(2)} \biguplus M^{(\geq 3)} \\ &= \#\big(M^{(1)} \setminus H\big) \biguplus U_{\leq e}^{(2)} \biguplus Z \biguplus [\mu', f) \\ &\geq \#\big(M^{(1)} \setminus H\big) \biguplus [\mu', f) \biguplus U_{\leq e}^{(2)} \biguplus U_{\leq e}^{(\geq 3)} \\ &= \#\{h \in U : h \leq_{\mathrm{dlex}} \mu'\}. \end{aligned}$$

Since μ' is admissible over U, this contradicts the choice of f. $\square$

Now

$$W = \{h \in U^{(2)} : h \leq_{\mathrm{dlex}} g'\} \biguplus Z$$

and since $W \gg Y$ and $Y = X \uplus \{h \in U^{(2)} : h \leq_{\mathrm{dlex}} g'\} \uplus M^{(\geq 3)}$, we have

$$m(Z) \succeq m\big(X \biguplus M^{(\geq 3)}\big) = m\big(H \biguplus M^{(\geq 3)}\big),$$

which proves (11).

6.3. **Proof of Proposition 6.3 when $f = \delta_1 x_1^{e-b_1-1}$.**

In this subsection, we prove Proposition 6.3 when $f = \delta_1 x_1^{e-b_1-1}$. Let $F = M_e^{(1)}$. If $F = \emptyset$ then there is nothing to prove. Thus we may assume $F \neq \emptyset$. Then $M \supset U_{\leq e}^{(\geq 2)}$ since M is a ladder set.

Case 1. Suppose $a_2 = 0$. Since $\delta_1 x_2^{e-b_1}$ is admissible over U, $\delta_1 x_2^{e-b_1} \notin F$. Indeed, if $\delta_1 x_2^{e-b_1} \in F$ then $M \supset \{h \in U : h \leq_{\mathrm{dlex}} \delta_1 x_2^{e-b_1}\}$, which contradicts the choice of f. Thus

$$F \subset [\delta_1 x_2^{e-b_1}, \delta_1 x_n^{e-b_1}].$$

and

$$\rho(F) \subset \rho\big([\delta_1 x_2^{e-b_1}, \delta_1 x_n^{e-b_1}]\big) = U_e^{(2)}.$$

Consider

$$X = \rho(F) \biguplus U_{\leq e-1}^{(2)} \biguplus M^{(\geq 3)} \subset U^{(\geq 2)}$$

and let $Y \subset U^{(\geq 2)}$ be the extremal set with $\#Y = \#X$. Since X is a ladder set in $U^{(\geq 2)}$, by the induction hypothesis we have

$$Y \gg X.$$

We claim

Lemma 6.10. $Y^{(2)} = U^{(2)}_{\le e-1}$.

Proof. Suppose contrary that $Y^{(2)} \ne U^{(2)}_{\le e-1}$. Let $g = \delta_2 \bar{g}$ be the largest admissible monomial in $Y^{(2)}_{\le e}$ over $U^{(\ge 2)}$ with respect to $>_{\mathrm{dlex}}$. Since $X \supset U^{(\ge 2)}_{\le e-1}$, we have $Y \supset U^{(2)}_{\le e-1}$ by Lemma 5.10. Thus $\deg g = e$ and $Y \supset U^{(\ge 3)}_{\le e}$.

Let $g' = \delta_1 \bar{g}$. Since $g = \delta_2 \bar{g}$ is admissible over $U^{(\ge 2)}$ and since $\rho(g') = g$, g' is admissible over U by Lemma 5.9. Observe $\#Y = \#X \le \#F + \#M^{(\ge 2)} - \#U^{(2)}_e$. Then

$$\begin{aligned}
\#M &\ge \#U^{(1)}_{\le e-1} + \#U^{(2)}_e + \#Y \\
&\ge \#U^{(1)}_{\le e-1} + \#U^{(2)}_e + \#\{h \in U^{(\ge 2)} : h \le_{\mathrm{dlex}} g\} \\
&= \#U^{(1)}_{\le e-1} + \#U^{(2)}_e + \#U^{(2)}_{\le e-1} \biguplus [g, \delta_2 x_n^{e-b_1}] \biguplus U^{(\ge 3)}_{\le e} \\
&= \#U^{(1)}_{\le e-1} + \#U^{(\ge 2)}_{\le e} + \#[g', \delta_1 x_n^{e-b_1}] \\
&= \#\{h \in U : h \le_{\mathrm{dlex}} g'\}
\end{aligned}$$

which contradicts the choice of f. Hence $Y^{(2)} = U^{(2)}_{\le e-1}$. □

Then, since $Y \gg X$, we have

$$Y^{(\ge 3)} \gg F \biguplus M^{(\ge 3)}. \tag{12}$$

Let

$$N = U^{(1)}_{\le e-1} \biguplus M^{(2)} \biguplus Y^{(\ge 3)}.$$

Then N is a ladder set since $\#Y^{(\ge 3)} \ge \#M^{(\ge 3)}$. Also $N \gg M$ by (12). Thus N satisfies the desired conditions.

Case 2. Suppose $a_2 > 0$. Since $\deg f \ne e$, we have $\#M < \#U^{(1)}_{\le e}$ by Lemma 5.12. Hence

$$\#F + \#M^{(2)} \le \#M - \#U^{(1)}_{\le e-1} < \#U^{(1)}_e \le \#U^{(2)}_{\le e+a_2}. \tag{13}$$

Then, by ($A2$) and ($A3$), we may assume that $\rho(F) \cap M^{(2)} = \emptyset$, $t \ge 3$ and there exists a $d \ge e$ such that $M^{(2)} = U^{(2)}_{\le d}$ and $M^{(3)}_{d+1} \ne U^{(3)}_{d+1}$.

Let

$$A = \big\{\delta_2 u \in \rho(F)_{e+a_2} : x_2^{(e+a_2)-(d+1)} \text{ divides } u \text{ and } \delta_2 u / x_2^{(e+a_2)-(d+1)} \notin \rho(F)_{d+1}\big\},$$

$$E = x_2^{-(e+a_2)+(d+1)} A \subset U^{(2)}_{d+1},$$

and

$$B = \rho(F)_{e+a_2} \setminus A \subset U^{(2)}_{e+a_2}.$$

(See the second picture in Fig. 21.)

(subcase 2-1) Suppose $\#B + \#M^{(\geq 3)} < \#U^{(2)}_{e+a_2}$. Consider

$$U' = U^{(2)} \biguplus \left(\biguplus_{i=3}^{t} U^{(i)}[-a_2] \right).$$

Since $M^{(\geq 3)}[-a_2] \supset U'^{(\geq 3)}_{\leq e+a_2}$, by Corollary 5.13 and the induction hypothesis, there exists the extremal set $Q \subset U'^{(\geq 3)}$ such that

$$Q \gg B \biguplus M^{(\geq 3)}. \tag{14}$$

Let P be the super rev-lex set in $U^{(2)}$ with $\#P = \#M^{(2)} + \#\rho(F) \setminus B$. Then since $\rho(F)_{\leq e+a_2-1} \biguplus E$ is rev-lex, Corollary 4.6 shows

$$m\big(M^{(2)} \biguplus \rho(F) \setminus B\big) = m\big(M^{(2)}\big) + m\big(\rho(F)_{\leq e+a_2-1} \biguplus E\big) \preceq m(P). \tag{15}$$

(See the second step in Fig. 21.) We claim that

$$N = U^{(1)}_{\leq e-1} \biguplus P \biguplus Q[+a_2] \subset U$$

satisfies the desired conditions. Indeed, by (14) and (15),

$$m(N) \succeq m\big(U^{(1)}_{\leq e-1} \biguplus M^{(2)} \biguplus \big(\rho(F) \setminus B\big) \biguplus \big(B \biguplus M^{(\geq 3)}\big)\big) = m(M).$$

(See Fig. 21). It remains to prove that N is a ladder set. If $\rho(F) \setminus B = \emptyset$ then $P = M^{(2)}$, and therefore N is a ladder set since $\#Q \geq \#M^{(\geq 3)}$. Suppose $\rho(F) \setminus B \neq \emptyset$. Recall that $\rho(F) \cap M^{(2)} = \emptyset$. Since

$$\#U^{(2)}_{\leq e} \leq \#M^{(2)} \leq \#P = \#\rho(F)_{\leq e+a_2-1} \biguplus E \biguplus M^{(2)} \leq \#U^{(2)}_{\leq e+a_2-1},$$

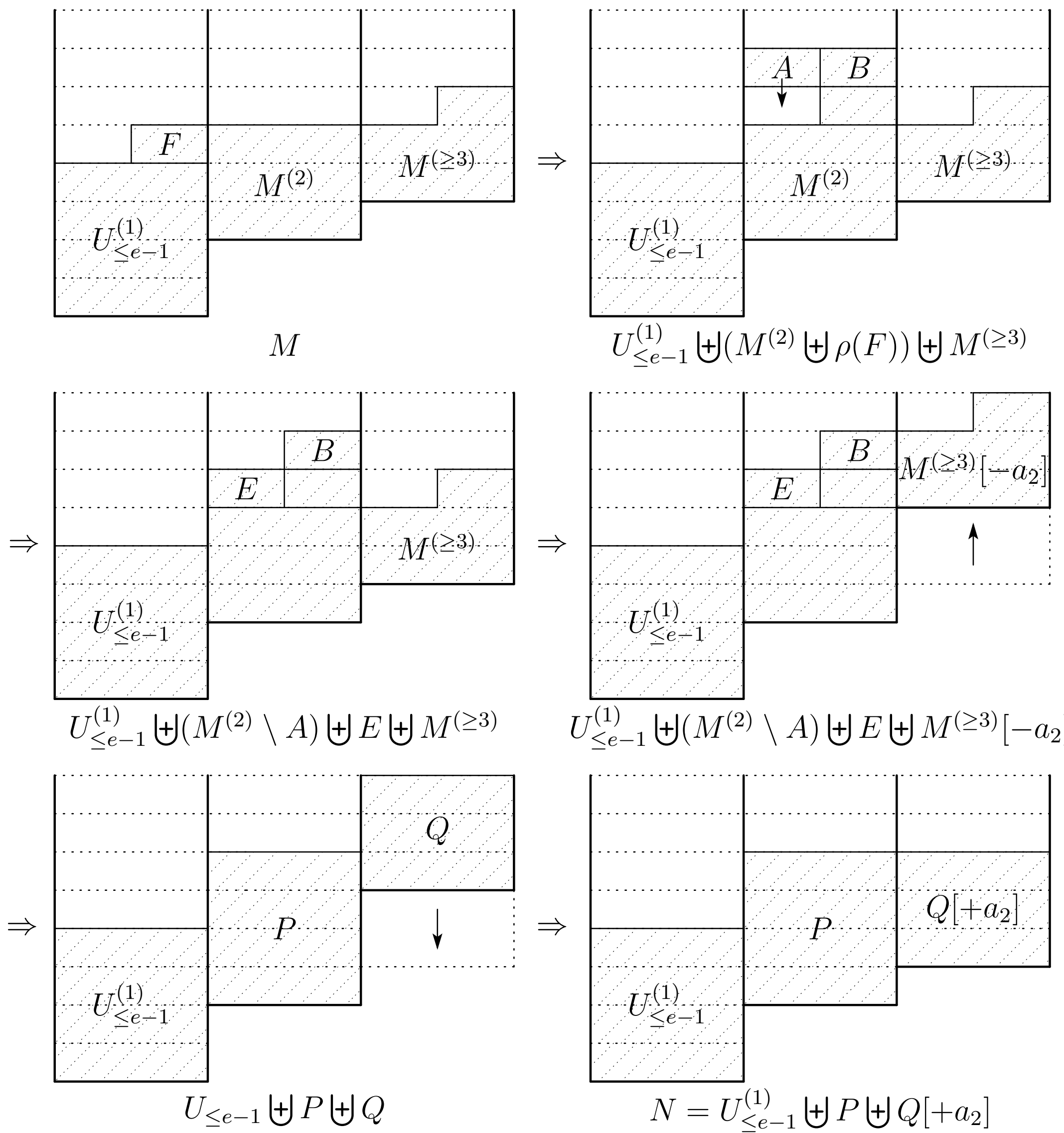
we have

$$U^{(2)}_{\leq e} \subset P \subset U^{(2)}_{\leq e+a_2-1}.$$

Then by Lemma 5.10 what we must prove is

$$\#Q \geq \#U^{(\geq 3)}_{\leq e+a_2-1}.$$

Figure 21



Since $\#S^{(i)}_k = \sum_{j=i}^n \#S^{(j)}_{k-1}$ for all $i > 0$ and $k > 0$, we have

$$\#U^{(3)}_k \geq \sum_{j=3}^{t} \#U^{(j)}_{k-1} = \#U^{(\geq 3)}_{k-1} \tag{16}$$

for all $k > 0$. Since $\rho(F) \setminus B \neq \emptyset$, $\#B = \#\rho(F)_{e+a_2} \setminus A \geq \#U^{(2)}_{e+a_2} - \#U^{(2)}_{d+1}$. Thus

$$\#B \geq \#U^{(2)}_{e+a_2} - \#U^{(2)}_{d+1} = \# \biguplus_{j=d+2}^{e+a_2} U^{(3)}_{j+a_3} \geq \# \biguplus_{j=d+2}^{e+a_2} U^{(3)}_j \geq \sum_{j=d+1}^{e+a_2-1} \#U^{(\geq 3)}_j,$$

(we use (16) for the last step) and therefore

$$\#Q = \#M^{(\geq 3)} + \#B \geq \#U^{(\geq 3)}_{\leq d} + \sum_{d+1}^{e+a_2-1} U^{(\geq 3)}_j \geq \#U^{(\geq 3)}_{\leq e+a_2-1}$$

as desired.

(subcase 2-2) Suppose $\#B + \#M^{(\geq 3)} \geq \#U^{(2)}_{e+a_2}$. We first prove

Lemma 6.11. $\rho(F) \not\supset \biguplus_{j=d+2}^{e+a_2} U^{(2)}_j$.

Proof. Suppose contrary that $\rho(F) \supset \biguplus_{j=d+2}^{e+a_2} U^{(2)}_j$. Then

$$\#\rho(F) \setminus B = \#\big(\rho(F) \setminus (A \biguplus B)\big) \biguplus E = \# \biguplus_{j=d+1}^{e+a_2-1} U^{(2)}_j$$

by the choice of E. Then

$$\#\big(\rho(F) \setminus B\big) \biguplus M^{(2)} \geq \#U^{(2)}_{\leq e+a_2-1}$$

and

$$\#M = \#U^{(1)}_{\leq e-1} + \#\rho(F) + \#M^{(\geq 2)} \geq \#U^{(1)}_{\leq e-1} + \#U^{(2)}_{\leq e+a_2-1} + \#U^{(2)}_{e+a_2} = \#U^{(1)}_{\leq e},$$

where we use the assumption $\#B+\#M^{(\geq 3)} \geq \#U^{(2)}_{e+a_2}$ for the second step. However, since $\deg f < e$ and $a_2 > 0$, Lemma 5.12 says

$$\#M < \#U^{(1)}_{\leq e},$$

a contradiction. □

The above lemma says that $e + a_2 \geq d + 2$ and $\rho(F)_{d+1} = \emptyset$. Thus B does not contain any monomial $\delta_2 u$ such that u is divisible by $x_2^{(e+a_2)-(d+1)}$. Hence

$$\rho(B) \subset \biguplus_{j=d+2+a_3}^{e+a_2+a_3} U^{(3)}_j.$$

Since $M^{(3)}_{d+1} \neq U^{(3)}_{d+1}$, by Lemma 5.15,

$$\#M^{(\geq 3)} < \#U^{(3)}_{\leq d+2}.$$

We claim

Lemma 6.12. $a_3 = 0$.

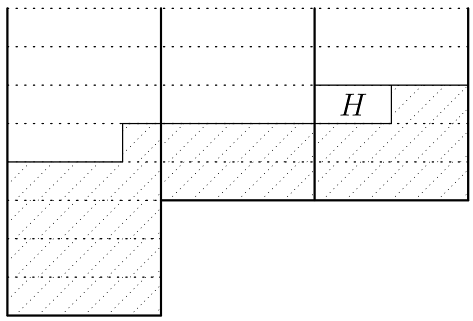
Proof. If $a_3 > 0$ then

$$\#B + \#M^{(\geq 3)} < \# \biguplus_{j=d+2+a_3}^{e+a_2+a_3} U^{(3)}_j + \#U^{(3)}_{\leq d+2} \leq U^{(3)}_{\leq e+a_2+a_3} = \#U^{(2)}_{e+a_2},$$

which contradicts the assumption of (subcase 2-2). □

Let
$$H=\{h\in U^{(\geq 3)}_{d+1} : h\not\in M^{(\geq 3)}\}.$$
(See Fig. 22.)

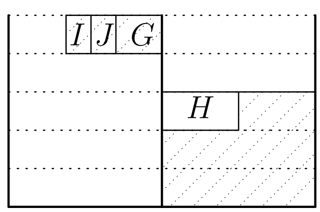
Figure 22



M

By Lemma 5.15,
$$\#H+\#M^{(\geq 3)}<\#U^{(3)}_{\leq d+2}.$$
Hence by the assumption of (subcase 2-2)
$$\#B\geq \#U^{(2)}_{e+a_2}-\#M^{(\geq 3)}=\#U^{(3)}_{\leq e+a_2}-\#M^{(\geq 3)}>\#H+\#\biguplus_{j=d+3}^{e+a_2}U^{(3)}_j.$$
Let
$$B=I\biguplus J\biguplus G$$
such that I is the set of lex-largest $\#H$ monomials in B and G is the rev-lex set with $\rho(G)=\biguplus_{j=d+3}^{e+a_2}U^{(3)}_j$. (See Fig. 23.)

Figure 23



$B\biguplus M^{(\geq 3)}$

Since $\rho(B)\subset \biguplus_{j=d+2}^{e+a_2}U^{(2)}_j$, $\rho(I)\subset U^{(3)}_{d+2}$. Let $C\subset U^{(3)}_{d+2}$ be the lex set in $U^{(3)}_{d+2}$ with $\#C=\#\rho(I)=\#H$. If we regard $U^{(\geq 3)}$ as an universal lex ideal in $K[x_3,\dots,x_n]$, then H and C are lex sets in $K[x_3,\dots,x_n]$ with the same cardinality. Hence $C=x_3H$. Then, by the interval lemma,
$$m(H)=m(C)\succeq m\big(\rho(I)\big)=m(I) \tag{17}$$

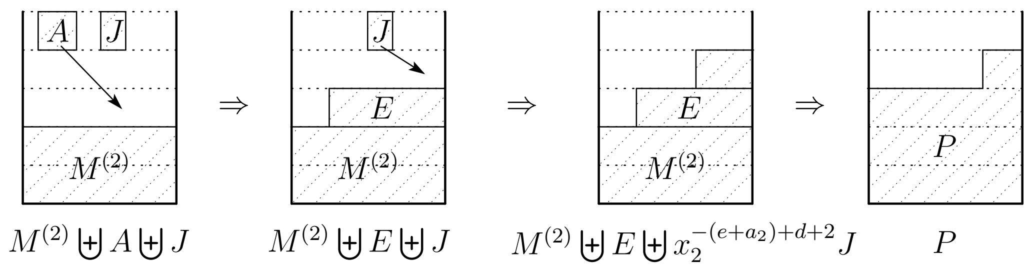
Let $P\subset U^{(2)}$ be the super rev-lex set with $\#P=\#A+\#J+\#M^{(2)}$. By the choice of G, G is the set of all monomials $\delta_2 u\in\rho(F)$ such that u is not divisible by

$x_2^{e+a_2-(d+2)}$. Also, since B does not contain any monomial $\delta_2 u$ such that u is divisible by $x_2^{e+a_2-(d+1)}$, any monomial in J is divisible by $\delta_2 x_2^{e+a_2-(d+2)}$. Then $x^{-(e+a_2)+d+2}J \subset U^{(2)}_{d+2}$ is a rev-lex set. Since $M^{(2)} \biguplus E \biguplus (x_2^{-(e+a_2)+(d+2)}J)$ is rev-lex,

$$m(P) \succeq m\big(M^{(2)} \biguplus E \biguplus x_2^{-(e+a_2)+(d+1)}J\big) = m\big(M^{(2)} \biguplus A \biguplus J\big). \tag{18}$$

(See Fig. 24.)

Figure 24



Let

$$Q = \rho(F) \setminus (A \biguplus B) = \rho(F)_{\le e+a_2-1}.$$

(subcase 2-2-a) Suppose that $\#P + \#Q \le \#U^{(2)}_{\le e+a_2-1}$. Let $R \subset U^{(2)}$ be the super rev-lex set with $\#R = \#P + \#Q$. Then since Q is an upper rev-lex set of degree $e + a_2 - 1$, by Corollary 4.5 and (18)

$$m(R) \succeq m\big(P \biguplus Q\big) \succeq \big(M^{(2)} \biguplus A \biguplus J \biguplus Q\big) \tag{19}$$

On the other hand, by Lemma 5.15,

$$\#H + \#M^{(\ge 3)} < \#U^{(3)}_{\le d+2}.$$

Then since $\rho(G) = \biguplus_{j=d+3}^{e+a_2} U^{(3)}_j$,

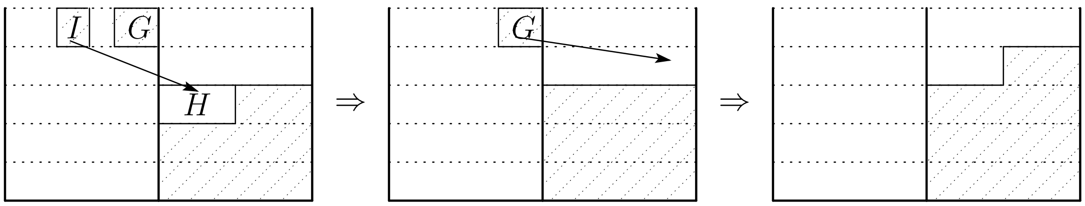
$$\#I \biguplus G \biguplus M^{(\ge 3)} = \#G \biguplus H \biguplus M^{(\ge 3)} < \#U^{(3)}_{\le e+a_2} = \#U^{(2)}_{e+a_2}.$$

Let $U' = U^{(2)} \biguplus (\biguplus_{i=3}^{t} U^{(i)}[-a_2])$. Observe that $M^{(3)}[-a_2] \supset U'^{(\ge 3)}_{\le e+a_2}$. Then Lemma 5.13 and (17) say that there exists an extremal set $Z \subset U^{(\ge 3)}[-a_2]$ such that

$$Z \gg G \biguplus H \biguplus M^{(\ge 3)} \gg G \biguplus I \biguplus M^{(\ge 3)} \tag{20}$$

(See Fig. 25.)

Figure 25



$I \biguplus G \biguplus M^{(\geq 3)}$ $\quad$ $G \biguplus H \biguplus M^{(\geq 3)}$ $\quad$ Z

We claim that

$$N = U^{(1)}_{\leq e-1} \biguplus R \biguplus Z$$

satisfies the desired conditions. Indeed, by (19) and (20),

$$\begin{aligned} N &\gg U^{(1)}_{\leq e-1} \biguplus \big(M^{(2)} \biguplus A \biguplus J \biguplus Q\big) \biguplus G \biguplus I \biguplus M^{(\geq 3)} \\ &\gg U^{(1)}_{\leq e-1} \biguplus F \biguplus M^{(2)} \biguplus M^{(\geq 3)} \\ &= M. \end{aligned}$$

(We use $\rho(F) = A \biguplus I \biguplus J \biguplus G \biguplus Q$ and $m(F) = m(\rho(F))$ for the second step.) It remains to prove that N is a ladder set. Since $U^{(2)}_{\leq d} \subset R \subset U^{(2)}_{\leq e+a_2-1}$ it is enough to prove that $Z \supset U^{(\geq 3)}_{\leq e+a_2-1}$. Since $\rho(G) = \biguplus_{j=d+3}^{e+a_2} U^{(3)}_j$,

$$\#Z = \#\big(H \biguplus M^{(\geq 3)} \biguplus G\big) \geq \#U^{(\geq 3)}_{\leq d+1} \biguplus \Big(\biguplus_{j=d+3}^{e+a_2} U^{(3)}_j \Big) \geq \#U^{(\geq 3)}_{\leq e+a_2-1}.$$

(We use $\#U^{(3)}_j \geq \#U^{(\geq 3)}_{j-1}$ for the last step.) Then $Z \supset U^{(\geq 3)}_{\leq e+a_2-1}$ by Lemma 5.10 as desired.

(subcase 2-2-b) Suppose that $\#P + \#Q > \#U^{(2)}_{\leq e+a_2-1}$. Note that

$$\#P + \#Q + \#I + \#G = \#F + \#M^{(2)}.$$

Then $\#M^{(2)} \biguplus F > \#U^{(2)}_{\leq e+a_2-1}$. Let R be the super rev-lex set with $\#R = \#M^{(2)} + \#F$. Then $\#R = \#M^{(2)} + \#F \leq \#U^{(2)}_{\leq e+a_2}$ by (13). Since $\#R \geq \#P + \#Q > U^{(2)}_{\leq e+a_2-1}$, there exists a rev-lex set $B' \subset U^{(2)}_{e+a_2}$ such that

$$R = U^{(2)}_{\leq e+a_2-1} \biguplus B'.$$

Also by Corollary 4.5,

$$B' \biguplus U^{(2)}_{\leq e+a_2-1} = R \gg M^{(2)} \biguplus \rho(F). \tag{21}$$

Since $\#F + \#M^{(\geq 2)} < \#U^{(2)}_{\leq e+a_2}$, we have $\#B' + \#M^{(\geq 3)} < \#U^{(2)}_{e+a_2}$. Then by Lemma 5.13 there exists the extremal set $Z \subset U^{(\geq 3)}[-a_2]$ such that

$$B' \biguplus M^{(\geq 3)}[-a_2] \ll Z. \tag{22}$$

We claim that

$$N = U^{(1)}_{\le e-1} \biguplus U^{(2)}_{\le e+a_2-1} \biguplus Z[+a_2]$$

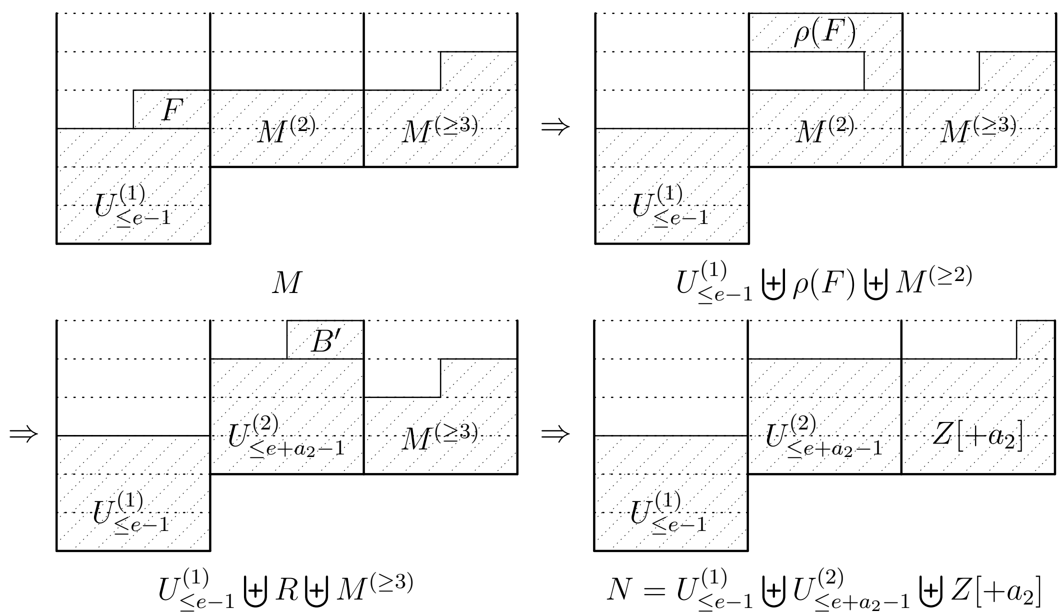
satisfies the desired conditions.

By (21) and (22),

$$\begin{aligned} N &\gg U^{(1)}_{\le e-1} \biguplus U^{(2)}_{\le e+a_2-1} \biguplus B' \biguplus M^{(\ge 3)} \\ &\gg U^{(1)}_{\le e-1} \biguplus F \biguplus M^{(2)} \biguplus M^{(\ge 3)} = M. \end{aligned}$$

(See Fig. 26.)

Figure 26



It remains to prove that N is a ladder set. What we must prove is

$$Z[+a_2] \supset U^{(\ge 3)}_{\le e+a_2-1}.$$

By the assumption of (subcase 2-2-b),

$$\#M^{(2)} + \#F - \#\big(I \biguplus G\big) = \#Q + \#P > \#U^{(2)}_{\le e+a_2-1}.$$

Then

$$\#B' = \#M^{(2)} + \#F - \#U^{(2)}_{\le e+a_2-1} > \#I \biguplus G.$$

Then in the same way as the computation of $\#Z$ in (subcase 2-2-a), we have

$$\#Z = \#M^{(\ge 3)} \biguplus B' \ge \#M^{(\ge 3)} \biguplus (I \biguplus G) \ge \#U^{(\ge 3)}_{\le e+a_2-1}.$$

Then by Lemma 5.10, $Z[+a_2] \supset U^{(\ge 3)}_{\le e+a_2-1}$ as desired.

7. Examples

In this section, we give some examples of saturated graded ideals which attain maximal Betti numbers for a fixed Hilbert polynomial. Observe that, by the decomposition given before Definition 3.7, the Hilbert polynomial of a proper universal lex ideal $I=(\delta_1,\delta_2,\dots,\delta_t)$ is given by

$$H_I(t)=\binom{t-b_1+n-1}{n-1}+\binom{t-b_2+n-2}{n-2}+\cdots+\binom{t-b_t+n-t}{n-t},$$

where $b_i=\deg\delta_i$ for $i=1,2,\dots,t$.

Example 7.1. Let $S=K[x_1,\dots,x_4]$ and $\bar{S}=K[x_1,\dots,x_3]$. Consider the ideal $I=(x_1^3,x_1^2x_2,x_1x_2^2,x_2^3,x_1^2x_3)\subset S$. Then

$$H_I(t)=\frac{1}{6}t^3+t^2-\frac{15}{6}t+1=\binom{t+2}{3}+\binom{t-4}{2}+\binom{t-9}{1}$$

and the proper universal lex ideal with the same Hilbert polynomial as I is

$$L=(x_1,x_2^6,x_2^5x_3^5).$$

Let

$$U=\operatorname{sat}\bar{L}=\left(\bar{L}:x_3^\infty\right)=(x_1,x_2^5)\subset\bar{S}$$

and $c=\dim_K U/\bar{L}=5$. Then the extremal set $M\subset U$ with $\#M=5$ is

$$M=x_1\{1,x_1,x_2,x_3\}\biguplus x_2^5\{1\}.$$

Then the ideal in S generated by all monomials in $U\setminus M$ is

$$J=x_1(x_1^2,x_1x_2,x_1x_3,x_2^2,x_2x_3,x_3^2)+x_2^5(x_2,x_3)\subset S,$$

and J has the largest total Betti numbers among all saturated graded ideals in S having the same Hilbert polynomial as I.

Example 7.2. Let $S=K[x_1,\dots,x_5]$ and $\bar{S}=K[x_1,\dots,x_4]$. Consider the ideal $I=(x_1,x_2^2,x_2x_3^3,x_2x_3^2x_4^{15})$. Then I is a proper universal lex ideal. Let

$$U=\operatorname{sat}\bar{I}=\left(\bar{I}:x_4^\infty\right)=(x_1,x_2^2,x_2x_3^2)\subset\bar{S}$$

and $c=\dim U/\bar{I}=15$. Then the extremal set $M\subset U$ with $\#M=15$ is

$$M=x_1\{1,x_1,x_2,x_3,x_4,x_2x_3,x_2x_4,x_3^2,x_3x_4,x_4^2\}\uplus x_2^2\{1,x_2,x_3,x_4\}\uplus x_2x_3^2\{1\}.$$

Then the ideal in S generated by all monomials in $U\setminus M$ is

$$\begin{aligned} J \;=\;& x_1(x_1^2,x_1x_2,x_1x_3,x_1x_4,x_2^2,x_2x_3^2,x_2x_3x_4,x_2x_4^2,x_3^3,x_3^2x_4,x_3x_4^2,x_4^3)\\ &+x_2^2(x_2^2,x_2x_3,x_2x_4,x_3^2,x_3x_4,x_4^2)+x_2x_3^2(x_3,x_4)\end{aligned}$$

and J has the largest total Betti numbers among all saturated graded ideals in S having the same Hilbert polynomial as I.

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